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A

PRACTICAL SYSTEM OF ALGEBRA,

IN

THEORY AND PRACTICE.

IN TWO PARTS.

WITH A

NEW METHOD OF

SOLVING CUBIC EQUATIONS AND THOSE OF THE HIGHER ORDERS.

WITH NOTES AND ILLUSTRATIONS.

INTENDED FOR THE USE OF SCHOOLS, ACADEMIES, AND PRIVATE STUDENTS.

PREPARED BY JOHN D. WILLIAMS,

Author of an Elementary Treatise on Algebra, 8vo, published by Hilliard, Gray, & Co., A Key to Hutton's Mathematics, containing the Questions and their Solutions, "Arithmetical and Algebraical Amusements," Arithmetic and Key, &c. &c.

SECOND EDITION.

ASTOR LIBRARY
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BOSTON:
PUBLISHED BY MARSH, CAPEN, LYON & WEBB.
FOR THE AUTHOR.
1840.

Entered according to act of Congress, in the year 1839, by John D. Williams, in the Office of the Clerk of the Southern District of New York.

Repair No. 126/03

PREFACE.

Algebra is allowed to be the grand pillar on which the whole science of Mathematics depends. It is the foundation on which the glorious superstructure of the abstruse sciences has been reared, and is therefore of undoubted importance and utility. Indeed, volumes have been written on the elegance and importance of this study, but the subject is inexhausted; and the author deems no apology necessary for dwelling a few moments on a subject which has occupied much of his time, and which has proved a source of the purest gratification while engaged in its pursuits.

We have said that Algebra is the foundation of all the abstruse sciences; the assertion is not made at random, but admits of ample proofs and clear demonstrations. Unlike arithmetic in this respect, it does not merely consist in understanding the common routine of mechanical and mercantile pursuits, but its study expands the intellect, enlarges the reasoning faculties, and accustoms the juvenile mind to patient attention and accurate reasoning. By its operations, the laws which govern the planetary system have been calculated, their order, harmony, and regularity have been displayed, and their general characteristics have been developed; the trackless ocean has been traversed, its boundaries have been determined, and a communication has been opened with every corner of our globe. Hence the increasing taste for these studies is readily accounted for, and we cease to wonder at the numerous compilations which have appeared, in order to facilitate the labors of the student. Unfortunately, however, the increasing demand for works of this nature has called into existence an ephemeral race of authors, who, instead of lessening the labors of the student, have strown his path with new difficulties, and harassed and perplexed him with unmeaning and tautological rules and expressions, insomuch that many have given up its pursuit on the very threshold, and others, having proceeded a short distance, and found new and increasing difficulties at every step, have closed the volume, and with it all attempts at mathematical studies, in hopeless despair.

It may be here mentioned, that some of these works on algebra are mere copies of others, and in one instance a person of this city, though, to our credit be it spoken, not an American, having republished a European work, claimed the merit of it as his own, after having *copied verbatim* from another, and with so little judgment that the *very errors* of the *press*, in the London edition, appear glaringly, and without comment, in the work which this individual (*James Ryan, N. Y.*) claimed as exclusively his own.

As it respects the method which should be adopted in pursuing the study of algebra, it should be remarked, that it is of primary importance that the student should in the first place make himself master of the common rules of arithmetic. Unless this is accomplished, it would be mere waste of time to attempt proceeding in the study of this science, which, although beautifully simple in its rules, is precise and accurate in its investigations. It is also the nature of all mathematical sciences, and of algebra in particular, to advance in continued progression, patiently but steadily; and hence the obvious necessity of the student's learning every thing thoroughly, and of perfectly understanding every rule and question as he advances, before he proceeds to another.

By these means the study, instead of being a toil, will become a pleasure, what at first appeared difficult will become easy, and the student will find new beauties allure him at every step, and cheer him through every difficulty. Let the diligent student also bear in mind that genius without application is useless, and that continued and untiring perseverance can accomplish almost every thing, however arduous it may appear at the outset. When, however, to a genius and a taste for mathematical pursuits is added a persevering industry, the juvenile mind ascends the ladder of science to the topmost round, and gazes eagle-eyed on that imperishable tablet where are recorded the names of a Newton, an Euler, Galileo, Leibnitz, Lagrange, and Laplace, who, by their works, have not only shed a lustre upon the science, but have gained for themselves an immortality that shall endure till time shall be no longer. Our own country too, though it is comparatively young, and has made but little progress in the mathematical sciences, has yet produced a few men who would have done honor to any age or country; and when the green sod of the valley shall have covered their mortal remains, when the present generation shall have passed away and been forgotten, their names will acquire unfading lustre, and be hailed by generations yet unborn, as the luminaries of science, and as the benefactors of the human race. Posterity must pass their eulogia, to posterity they must look for their fame and their immortality. And here, could I find suitable language, I might pay a passing tribute to the memory of him who sleeps beneath the waves of the Atlantic, the young, the accomplished, the lamented Fisher.* But what avails it? He hath passed from his sphere of usefulness; his bright and glorious career is finished; but his memory yet lives, and will be cherished in the hearts of his countrymen as a legacy never to be forgotten.

It only remains now for the author to state the reasons which have induced him to this compilation, and the manner in which it has been treated.

To make an excuse, or to offer an apology for a work of this nature, is, I believe, unnecessary, because, if it should prove unworthy of the public patronage, no excuse will palliate its defects, and no prefatorial apology will be received in extenuation for accidental or wilful error. Hence, apology is needless; but the author believes it to be a duty he owes to the public to state the reasons which have caused this compilation, and they are briefly these: In his own opinion, and in that of some of the best mathematicians of our city, the different algebras at present in use in this country are defective in many particulars. The rules, it is believed, are generally not the best that could be given, and in some cases are tediously abstruse and perplexing; and though there is generally an ample sufficiency of theory, yet there is not practice enough to engage without wearying the attention of the student, and to excite without overburdening his reasoning faculties. These objections it has been the aim of the author to remove, and to treat the subject on a clear and rational foundation, with as much simplicity as possible, and, instead of making a mystery of the science, and putting new and imaginary difficulties in the way of the student, to remove them by every possible method.

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* He was a passenger in the packet-ship Albion, lost on the coast of Ireland.

For these reasons the questions are worked out at full length, and every thing explained in the operation; and it is believed that the time is gone by when the best method of teaching algebra was thought to be by giving the student a difficult question, and leaving him to ponder and pore for weeks over a set of, to him, unmeaning symbols and figures.

The solutions of the greatest part of the difficult questions, being all given, will be of utility, also, in an economical point of view; as to all the algebras now in use, Keys, containing solutions, have been published, and by the necessity of purchasing them the student is subjected to additional expense.

The student will find a greater number of new equations than have ever been published in any treatise before. As they are problems which possess a degree of interest to the mind of the young student, it has been thought they would prove highly useful by blending amusement with study. The method of solving the irreducible case of cubic equations is new and of great importance, as it solves the most difficult questions with the greatest ease.

The author, in conclusion, would take this opportunity of returning his grateful thanks to the public for the liberal patronage bestowed on his former publications; and in offering this new work to their notice, he fearlessly depends on the candor and impartiality of an enlightened and liberal public.

JOHN D. WILLIAMS.

Dighton, March 1, 1829.

Given $x^4+x^3-5x+25$, to find the value of x true to abo
figures, and also $x^3-x^2-2x=-1$, to find x.

```
                                                               —1
1      0       —5         25(2.236067977           — 2  —1(1·8019
2      6       12         14                       + 0  —2         7
3      6        7         11                       — 2   1
2     10       32          9.0736                   +1        .992
5     16       39          1.9264           2.8 ... 2.24         8
2     14        6.368   1.59570141                  1.24       412
7     30       45.368     33069859                   64        387
2      1.84     6.744     32697893          4.401 .....4401    371
9.2   31.84    52.112       371966              4.124401        15
 2     1.88     1.078047  328311                       1        12
9.4   33.72    53.190047     43654          4.40|39....3964      3
 2     1.92     1.086921     38305              4.132767         2
9.6   35.64    54.276968      4350          4|.4|0573 ...132     –
 2       2949     219521      4925              4.1368|63
9.83  35.9349  54.49648(9      425                    31
 3       2958     21988        383              4|.1|3|7(0
9.86  36.2307  54.716|37        42
 3       2967       2|20        38
9.89  36.5274  54.718|57
 3        595       2|2
|9.9|26 36.586|9  5|4|.7|2|1   See Rules that follow page 11
           60|
       36.64|6
       36|70|
```

All the possible forms of Cubic Equations exhibited at one vi

1. Given $x^3+ax^2=r$, or $x^3+x^2=500$.
2. Given $x^3-ax^2=r$, or $x^3-3x^2=5$.
3. Given $x^3-ax^2=-r$, or $x^3-48x^2=-200$.
4. Given $x^3+ax=r$, or $x^3+9x=6$.
5. Given $x^3-ax=r$, or $x^3-27x=36$.
6. Given $x^3-ax=-r$, or $x^3-12x=-12$.
7. Given $x^3+ax^2+bx=r$, or $x^3+5x^2+29x=1829$.
8. Given $x^3+ax^2-bx=r$, or $x^3+2x^2-3x=9$.
9. Given $x^3-ax^2+bx=r$, or $x^3-39,6x^2+585,6x=293$
10. Given $x^3-ax^2-bx=*\,r$, or $x^3-120x^2-300x=$
11. Given $x^3-ax^2-bx=-r$, or $x^3-x^2-2x=-1$.
12. Given $x^3-ax^2+bx=-r$, or $x^3-5x^2+2x=-1$
13. Given $x^3+ax^2-bx=-r$, or $x^3+2x^2-23x=-7$

THE

ELEMENTS OF ALGEBRA.

DEFINITIONS AND NOTATION.

1. **Algebra** is a general method of computation, in which abstract quantities are represented by letters, and their connexion pointed out by means of certain indeterminate characters or symbols, which have been invented for this purpose.

2. The leading rules in Algebra are the same as those in Arithmetic, viz: Notation, Addition, Subtraction, Multiplication and Division.

3. Known quantities are usually expressed by the first letters of the alphabet, a, b, c, &c., and unknown quantities by the last, x, y, z, &c., and this must be always understood, unless the contrary be expressed.

4. A simple equation, or an equation of the first degree, is that which contains the unknown quantity simply; that is, without any of its powers except the first. A quadratic equation, or an equation of the second degree, is that which contains the square, but no higher power, of the unknown quantity.

5. The sign $>$ is called *greater than*, and the sign $<$ is called *less than;* and the expression in which either of these signs occurs is called an inequality; thus, the inequality $a > b$ denotes that a is greater than b, and the inequality $a < b$ denotes that a is less than b; the greatest quantity being always placed at the opening of the sign.

6. The sign Plus, or $+$, denotes addition, and means that the quantities between which it is placed are to be added together; thus, $a + b$ means that the quantity represented by b is to be added to the quantity represented by a, and is read, a plus b. If a represents 9, and b 5, then $a + b$ represents 14.

7. This character — (or minus) is the sign of subtraction, and means that the quantity to which it is prefixed is to be taken from the former. If a represents 9, and b 5, then $a - b$ represents 4.

8. The sign $\times$ is called the sign of multiplication, and placed between two quantities, denotes that they are to be multiplied together. A point is often used instead of this sign, or, when the quantities to be multiplied together are represented by letters in the form of a word, the sign may be altogether omitted; thus,

$2 \times 4 \times 5$, or 2.4.5, is the continued product of 2, 4 and 5. Likewise, $7 \times a \times b$, or $7.a.b.$, or $7ab$, is the continued product of 7, a, and b.

9. $\div$ divided by. This character is the sign of division, and signifies that the former of the quantities between which it is placed is to be divided by the latter; thus, $a \div b$ means that the quantity a is to be divided by b. The division of one quantity by another is frequently represented by placing the dividend over the divisor, with a line between them, in which case the expression is called a fraction; thus, $\frac{\text{Dividend,}}{\text{Divisor,}}\ \frac{a}{b}$, signifies a divided by b; then a is the numerator and b the denominator of the fraction.

10. A quantity in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus, $a^{-1}, a^{-2}, a^{-3}, a^{-n}$, signify $\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^n}$, respectively; these are called the negative powers of a.

11. Points are generally made use of to denote proportion; thus, $a : b :: c : d$, signifies that a bears the same proportion to b that c bears to d.

$=$ equal to. This sign means that the quantities between which it is placed are equal to each other; $ax - by = cd + ad$, signifies that the quantity $ax - by$ is equal to the quantity $cd + ad$.

12. The sign $\backsim$ between two quantities means or signifies their difference; thus, $a \backsim x$, is $a \backsim x$ or $x \backsim a$, according as a or x is the greater; $a \overset{\backsim}{+} x$ signifies the sum or difference of a and x.

13. A vinculum ——, is a line drawn over several quantities, and signifies that the terms under it are to be taken as one whole, and to be affected with the same operation. The modern method of expressing the same thing is by the parenthesis ( ) or bracket, [ ]. Thus $(a + b) \times x$, $\overline{a+b} \times x$, or $[a + b] \times x$, means that the quantity represented by $a + b$ is to be multiplied by the quantity represented by x. Let $a = 3$, $b = 4$, then $(a + b) \times x = 7x$, and if $x = 1$, then $7x = 7$.

14. The powers of algebraic quantities are denoted by placing a small figure, called the index or (indices) exponent of the power, to the right hand of the letter, as a^2, a^3, a^4, &c.; so when the index is a fraction the numerator shows the power to which the quantity is first to be raised, and the denominator expresses the root to be extracted; thus $a^{\frac{2}{3}}$ denotes the cube root of a square; $a^{\frac{3}{2}}$ is the square root of a^3; and $a^{\frac{1}{2}}$ denotes the square root of a, the same as $\sqrt{a}$; $a^{\frac{1}{3}}$, the cube root of a, the same as $\sqrt[3]{a}$; $a^{\frac{3}{5}}$ the fifth root of

the cube of a, or $\sqrt[5]{a^3}$; and $a^{\frac{5}{9}}$ is the ninth root of a fifth power of a, or $\sqrt[9]{a^5}$; and $(a + b)^{\frac{1}{n}}$ denotes the nth root of the sum of a and b.

15. A surd, or irrational quantity, is that of which the value cannot be accurately expressed in numbers; thus 2, 3, 5, and 7, the root of quantities, are denoted by the radical $\sqrt{\ }$, with the proper index in it. The signs $\sqrt{\ }$, or $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, &c., are used to express the square, cube, biquadrate, &c., roots of the quantities before which they are placed; thus, $\sqrt{2}$, $\sqrt[2]{a}$, $\sqrt[3]{a^3} = a$, $\sqrt[4]{a^4} = a$.

16. The reciprocal of any quantity is that quantity inverted, or unity divided by it; so the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

17. A rational quantity is that which has no radical sign or index annexed to it, as a, $\frac{3}{5}a$, or $8a$.

18. The words *therefore*, *consequently*, or *that is*, are usually expressed by the symbol $\therefore$; thus the sentence therefore $a + b$ is equal to $c + d$, is expressed by $\therefore a + b = c + d$.

19. Positive or affirmative quantities are those which are to be added, as a or $+a$, or $3ax$, or $+ 3ax$. See ex. 1, p. 5. For when a quantity is found without a sign it is understood to be positive, or to have the sign $+$ prefixed, that is, always when it is a leading quantity; and a quantity without any coefficient is supposed to have one or unity before it; thus, $a =$ once a, ($1a$.)

20. Negative quantities are those which are to be subtracted, as Ex. 11, viz: $-2by^2, -6by^2, -by^2, -8by^2, -by^2$.

21. When two quantities are multiplied together, each, considered separately, is termed the coefficient of the other; but when one of them is a known quantity, it alone is termed the coefficient. Thus in the quantity ab, a is the coefficient of b, and b of a; but in $4x$, the numeral 4 is the coefficient of x, and x is never termed the coefficient of 4. Sometimes the coefficient is a compound quantity, as $(a + b)x$, or $ax + bx$; here the coefficient is $a + b$.

22. Like signs are such as are all positive, ($+$), plus, or all negative, ($-$), minus.

23. Unlike signs are when some are positive, plus, ($+$), and others minus, ($-$).

24. Like quantities are such as do not differ except in their coefficients; as $2ax^2$, and $-3ax^2$. Unlike quantities are those which differ in letters or power, as 2, a and b, or $6a^{\frac{1}{2}}x^{\frac{1}{2}}$, and $4a^2y^2$.

25. A monomial, or simple quantity, is a quantity consisting of one term only, as a; $3bx$; $-2xy$; $+3xy$.

26. A binomial quantity consists of two terms, as $a + x$, or $a - b$, the latter of which, being connected by the sign $-$, is sometimes called a residual.

27. A trinomial consists of three, as $a + bx - 3y$.

28. A quadrinomial is a quantity consisting of four terms, as $ax^3 + 3xy - 3y + bxy^2$, or $a + b - x + y$.

29. A multinomial, or polynomial, is a quantity consisting of many terms, or of an indefinite number of terms, as $a + b - c + d - x + y$, &c.

30. When two equal quantities are compounded with the sign $=$ between them, the comparison is called an equation; thus $3 + 4 = 7$, or $2x + 4 = 8$, is an equation; the quantity on the left hand side of the sign $=$ is called the first member of an equation, and the quantity on the right hand of it the second. In the equation $2x + b = a$; $2x + b$ is the first member, and a is the second. When the quantities which compose a number are separated by the signs $+$ or $-$, each quantity, so separated, is called a term; thus, the first member of the equation $2x + b = a$, contains two terms, viz. $2x$ and b; also a and b are the terms of ab or $\frac{b}{a}$, and a, b, x, y are the terms of the proportion $a : b :: x : y$

31. A quantity is said to be a multiple of another when it contains it a certain number of times exactly; thus $16a$ is a multiple of $4a$, as it contains it exactly four times.

32. An equation of the third, fourth, &c., degree, is one in which the highest power of the unknown quantity is the third, fourth, &c., powers; and in general, an equation, in which the unknown quantity is called an equation of the mth degree, and each of the two members of an equation is called a side.

33. If equal quantities be either increased or diminished by the same quantity, the results will be equal, or, in other words, if each side of an equation be either increased or diminished by the same quantity, the result will be an equation.

34. If each side of an equation be either multiplied or divided by the same quantity, the result will be an equation.

35. If each side of an equation be either involved to the same power, or evolved to the same root, the result will be an equation.

36. Hence, generally, whatever operations be performed on one side of an equation, if the same operations be performed on the other side, the result will be an equation.

37. Notation in Algebra is the method of representing any proposed quantity by means of certain symbols.

1. What is the numeral value of $a + b - c$, supposing $a = 7$, $b = 5$, and $c = 8$?

Here $a + b - c = 7 + 5 - 8 = 12 - 8 = 4$, Ans.

2. What is the value of $\frac{ax + by}{b + x}$, where $a = 4$; $b = 6$; $x = 3$, and $y = 7$?

Here $ax + by$, or $4 \times 3 + 6 \times 7 = 12 + 42 = 54$.

And $b + x$, or $6 + 3 = 9$. Hence $\frac{ax+by}{b + x} = \frac{54}{9} = 6$, Ans.

3. What is the numeral value of $\frac{ab}{c} + \frac{ax - d^3}{b} - \sqrt{ab^2}$, supposing $a = 9$; $b = 4$; $c = 3$; $d = 2$; and $x = 8$?

$$\frac{ab}{c} + \frac{ax - d^3}{b} - \sqrt{ab^2} = \frac{(9 \times 4)}{3} + \frac{(9 \times 8) - 2^3}{4} - \sqrt{(9 \times 4 \times 4)} = 16.$$

Here, ab, or $9 \times 4 = 36$, which, divided by c or 3, gives 12, the value of the first term. Then ax, or $9 \times 8 = 72$, from which subtracting d^3 or 8, there remains 64; which, divided by b, or 4, gives 16, the value of the second term. Therefore the sum of the first and second terms is 28. Then ab^2, or $9 \times 4 \times 4 = 144$, the square root of which is 12, the value of the third term; and this subtracted from the sum of the former terms, because connected by the sign $-$, gives 16, the value of the whole expression.

ADDITION.

38. Addition in Algebra is the method of finding the sum of several algebraic quantities, and connecting them together by their proper signs. This rule is generally divided into three cases.

CASE I. *To add like quantities with like signs.*

RULE. Add the coefficients of the several quantities together, and to their sum prefix the common signs and annex the common letter or letters.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.	Ex. 6.
a	$7a$	$-4ax$	$5ax$	$6x + 3y$	$4x + 7y$
$3a$	$2a$	$-2ax$	$9ax$	$2x + 5y$	$x + y$
$4a$	$9a$	$-6ax$	$14ax$	$8x + 8y$	$5x + 8y$

Ex. 7.	Ex. 8.	Ex. 9.	Ex. 10.	Ex. 11.
$3a$	$-3ax$	$2ay$	$2b + 3y$	$-2by^2$
$5a$	$-6ax$	$5ay$	$5b + 7y$	$-6by^2$
a	$-ax$	$4ay$	$b + 2y$	$-by^2$
$7a$	$-2ax$	$7ay$	$8b + y$	$-8by^2$
$12a$	$-7ax$	$16ay$	$4b + 4y$	$-by^2$
$28a$	$-19ax$	$34ay$	$20b + 17y$	$-18by^2$

Case II. *To add like quantities with unlike or different signs.*

Rule. Add all the positive or plus quantities into one sum, and all the negative or minus into another sum; subtract the less of these sums from the greater; to their difference prefix the sign of the greater sum, whether $+$ or $-$, and annex the common letter or letters.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.
$+5a$	$-5x$	$+3a$	$+9y$	$-6y\sqrt{x}$
$-3a$	$+3x$	$-2a$	$-6y$	$+3y\sqrt{x}$
$+2a$ sum.	$-2x$ sum.	$+1a$ sum.	$+3y$ sum.	$-3y\sqrt{x}$ sum.

Ex. 6.	Ex. 7.	Ex. 8.	Ex .9	Ex. 10.
$-3a$	$2a-3r^2$	$6r+5ay$	$-2a^2$	$3ay-7$
$+7a$	$-7a+5r^2$	$-3r+2ay$	$-3a^2$	$-ay+8$
$+8a$	$-3a+r^2$	$r-6ay$	$-8a^2$	$+2ay-9$
$-a$	$+a-3r^2$	$2r+ay$	$+10a^2$	$-3ay-11$
			$+13a^2$	$+12ay+13$
$+11a$	$-7a$ *	$6r+2ay$	$+10a^2$	$+13ay-6$

Note. In the 6th example, the sum of the positive or plus quantities exceeds the sum of the negative by $11a$; consequently the sign is $+$, according to the rule. In the 7th example, the sum of the positive or $+$ (plus) quantities is less by $7a$ than the sum of the negative or $-$ (minus) quantities; consequently, the sign is $-$, according to the rule. In the 9th example, the sum of the positive terms is $23a^2$, and the sum of the negative ones is $-13a^2$; their difference, therefore, is $+10a^2$, which is the sum required. The other examples are wrought in a similar manner. If the positive and negative quantities be equal, the sum is nothing, and they are said to destroy each other. See example 7, right hand column.

Ex. 11.	Ex. 12.	Ex. 13.	Ex. 14.	Ex. 15.
$+9x$	$-4a^2$	$-6a\sqrt{x}$	$+6ax$	$+6a+4x$
$-6x$	$+a^2$	$+4a\sqrt{x}$	$-2ax$	$+4a+8x$
$+7x$	$-9a^2$	$+a\sqrt{x}$	$+7ax$	$-5a-2x$
$-x$	$+7a^2$	$-5a\sqrt{x}$	$-ax$	$-7a-3x$
$+9x$	$-5a^2$	$-6a\sqrt{x}$	$+10ax$	$-2a+7x$

Ex. 16	Ex. 1.7	Ex. 18	
$-6x+4x^2-8$	$7\sqrt{y}-4(a+b)$	$a(a+b)$	$-3\sqrt{(a-x)}$
$+8x+x^2+1$	$6\sqrt{y}+2(a+b)$	$-4a(a+b)$	$+7\sqrt{(a-x)}$
$+5x-3x^2+9$	$2\sqrt{y}+(a+b)$	$-2a(a+b)$	$-8\sqrt{(a-x)}$
$+7x-5x^2-7$	$\sqrt{y}-3(a+b)$	$5a(a+b)$	$+14\sqrt{(a-x)}$
$14x-3x^2-5$	$16\sqrt{y}-4(a+b)$	*	$+10\sqrt{(a-x)}$

III. *To add quantities when some are like and others* : *or when all the quantities are unlike.*

:. Add the like quantities together, according to cases 1 connect the unlike quantities *in any order*, with their signs and coefficients prefixed.

:. 1.

$$\begin{array}{rccc}
 & \text{Ex. 2.} & \text{Ex. 3.} & \text{Ex. 4.} \\
+\ 4 & -3ax^{\frac{1}{2}} & +10\sqrt{ax} & +3y+4ax^{\frac{1}{2}} \\
+\ 12 & +\ ax^{\frac{1}{2}} & -\ 3\sqrt{ax} & -\ y-5ax^{\frac{1}{2}} \\
-\ 14 & +5ax^{\frac{1}{2}} & +\ 4\sqrt{ax} & +4y+2ax^{\frac{1}{2}} \\
+\ 3 & -6ax^{\frac{1}{2}} & -12\sqrt{ax} & -2y+6ax^{\frac{1}{2}} \\
\hline
-\ 5 & -3\sqrt{a^2x}=-3ax^{\frac{1}{2}} & -\ \sqrt{ax} & 4y+7ax^{\frac{1}{2}}
\end{array}$$

Ex. 5.

$$\frac{a}{b}-\frac{3c^2}{a}+\frac{7\sqrt{bc}}{x}-9\left(\frac{ab+x}{d}\right)$$

$$\frac{a}{b}+\frac{7c^2}{a}-\frac{12\sqrt{bc}}{x}+6\left(\frac{ab+x}{d}\right)$$

$$\frac{a}{\;}+\frac{4c^2}{a}-\frac{5\sqrt{bc}}{x}-3\left(\frac{ab+x}{d}\right)$$

Ex. 6.

$$\begin{array}{r}
3a^2+4bc-\ e^2+10 \\
-5a^2+6bc+\ 2e^2-15 \\
-4a^2-9bc-10e^2+21 \\
\hline
-6a^2+\ bc-\ 9e^2+16
\end{array}$$

$$\begin{array}{cccc}
7. & \text{Ex. 8.} & \text{Ex. 9.} & \text{Ex. 10.} \\
y & 2ax\ -30 & 2xy-2x^2 & 6xy-12x^2 \\
x & 3x^2-\ 2ax & 3x^2+\ xy & 3xy-\ 4x^2 \\
y & 5x^2-\ 3x^{\frac{1}{2}} & x^2+\ xy & 2xy+\ 4x^2 \\
x & 3\sqrt{x}+10 & 4x^2-3xy & 3xy+\ 4x^2 \\
\hline
y & 8x^2\ -20 & 6x^2+\ xy & 14xy-\ 8x^2
\end{array}$$

:. When quantities with literal coefficients are to be added r, it may be done by the coefficients, with oper signs, under a m, or between brackets, and then subjoining the common / to the sum or difference thus arising, as in the margin.

$$\begin{array}{cc}
ax+b & ax^2+bx \\
cx+d & cx^2-dx \\
\hline
(a+c)x+b+d & (a+c)x^2+(b-d)x
\end{array}$$

$$\begin{array}{cc}
\text{Ex. 1.} & \text{Ex. 2.} \\
ax+by^2 & x^2+adz \\
cdx+ady^2 & \frac{1}{2}x^2-nz \\
bx-cy^2 & bx^2+cez \\
 & dx^2-mz \\
\hline
-b)x+(b+ad-c)y^2 & (d+b+1\frac{1}{2})x^2+(ad-n+ce-m)z
\end{array}$$

equired the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$. Answer, a.

dd $5x-3a+b+7$, and $-4a-3x+2b-9$, together.

3. Add $2a+3b-4c-9$, and $5a-3b+2c-10$, together.
4. Add x^3+ax^2+bx+2, and x^3+cx^2+dx-1, together.

SUBTRACTION.

39. **Subtraction** in Algebra is the method of finding the difference between two algebraical quantities, and connecting those quantities together with their proper signs.

Rule 1. Set those quantities from which the subtraction is to be made in one line, and the quantities to be subtracted in a line below them, observing to place like quantities under each other when they occur.

Rule 2. Subtraction in Algebra is performed by simply changing the signs ($+$ into $-$, and $-$ into $+$) of the lower line of quantities to be subtracted, and then adding or connecting them as in addition, and the result will be the difference required.

	Ex. 1.			Ex. 2.
From	$8a$	or changing the sign of $3a$	To $8a$	From $+6a+5y-7$
Take	$3a$		Add $-3a$	Take $-3a+2y-3$
The answer or remainder,			$+5a$	$+9a+3y-4$

Here the quantity to be subtracted is $-3a+2y-3$; we therefore change its signs according to the 2d rule, and it becomes $+3a-2y+3$, and this added to the other quantity gives the remainder $+9a+3y-4$.

Ex. 3.	Ex. 4.	Ex. 5.	Ex. 6.
$5a^2-2b$	x^2-2y+3	$5xy+8x-2$	$+8y^2-2y-5$
$2a^2+5b$	$4x^2+9y-5$	$3xy-8x-7$	$-y^2+3y+2$
$3a^2-7b$	$-3x^2-11y+8$	$2xy+16x+5$	$+9y^2-5y-7$

EXAMPLES WITH LITERAL COEFFICIENTS.

Note. When the quantities that are to be subtracted have literal coefficients, the operation may be performed by placing the coefficients, with their proper signs, between brackets, as in addition, and then subjoining the common quantity, the same as in the margin.

$$\begin{array}{c}ax-b\\cx-d\\\hline(a-c)x+d-b\end{array}\qquad\begin{array}{c}ax^2+bx\\cx^2-cx\\\hline(a-c)x^2+(b+c)x\end{array}$$

Ex. 19. From $pxy+qxz-rz^2+s$
Take $mxy-pqxz-nz^2+a$
Remainder, $(p-m)xy+(1+p)\,qxz+(n-r)z^2+s-a$.

1. Required the difference of $\frac{a+b}{2}$, and $\frac{a-b}{2}$. Answer, b.

2. From $ax^3-bx^2+cx-5d$, take $bx^2+ex-12d$.

MULTIPLICATION.

40. Multiplication in Algebra is the method of finding the product of *two* or *more* indeterminate algebraic quantities, and is generally divided into three *cases*.

Case I. *When both factors are simple quantities.*

Rule. Multiply the coefficients of the two quantities together, and annex to the result (or product) all the letters in both factors, which will give the whole product required. If the signs of the factors be like, that is, both + or both —, the sign of the product is +; but if they are unlike, or one of them — and the other +, the sign of the product is —: and this is commonly expressed by saying, like signs give plus, and unlike signs minus.

	Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.
Multiply	$4a$	$4abc$	$9xy^2$	$-3abc$	$-6a^2bc$
by	$3b$	$3ac$	$-2x$	$5a^2b$	$-2b^2x^2$
	$12ab$	$12a^2bc^2$	$-18x^2y^2$	$-15a^3b^2c$	$+12a^2b^3cx^2$

Case II. *When one factor is a compound and the other a simple.* **Rule.** Multiply each or every term of the compound factor separately by the simple factor, and to each product prefix its proper sign, and the result will be the whole product.

Ex. 6.	Ex. 7 .	Ex.9 .	Ex.10 .
$3a-2b$	$6xy-8$	$12x-ab$	a^2-2x+1
$4a$	$3x$	$5a$	$4x$
$12a^2-8ab$	$18x^2y-24x$	$60ax-5a^2b$	$4a^2x-8x^2+4x$

Ex. 11.	Ex. 12.	Ex. 13.
$35x-7a$	$12xy-ax+6$	$31xy^2-4\sqrt{x}+a$
$-2x$	$3xy$	$-2\sqrt{b}$
$-70x^2+14ax$	$36x^2y^2-3ax^2y+18xy$	$-62xy^2\sqrt{b}+8\sqrt{bx}-2a\sqrt{b}$

Case III. *When both factors are compound quantities.*

Rule. Multiply every part of the multiplicand by each part of the multiplier, placing the products one after the other, with their proper signs; then add the several products together, as in common multiplication.

Ex.1.		Ex. 2.	Ex. 3.
Multiply	$a+b$	$a+b$	a^2+ab+b^2
by	$a+b$	$a-b$	$a-b$
1st, by a ··	a^2+ab	a^2+ab	$a^3+a^2b+ab^2$
2d, by b ··	$ab+b^2$	$-ab+b^2$	$-a^2b-ab^2-b^3$
Product	$a^2+2ab+b^2$	$a^2 \quad * \quad -b^2$	$a^3 \quad * \quad * \quad -b^3$

When we have two or more quantities to multiply together, it is indifferent which two we begin with; for the products will always be the same, as will appear from the following example.

Let it be proposed to find the value or product of the four following factors, viz:

$$\begin{array}{cccc} \text{(I.)} & \text{(II.)} & \text{(III.)} & \text{(IV.)} \\ (a+b) & (a^2+ab+b^2) & (a-b) & \text{and } (a^2-ab+b^2) \end{array}$$

1st. Multiply the factors I. and II. Next the factors III. and IV.

$$\begin{array}{l} a^2+ab+b^2 \\ a+b \\ \hline a^3+a^2b+ab^2 \\ \quad +a^2b+ab^2+b^3 \\ \hline a^3+2a^2b+2ab^2+b^3 \end{array} \qquad\qquad \begin{array}{l} a^2-ab+b^2 \\ a-b \\ \hline a^3-a^2b+ab^2 \\ \quad -a^2b+ab^2-b^3 \\ \hline a^3-2a^2b+2ab^2-b^3 \end{array}$$

It remains now to multiply the first product I. II. by the second product III. IV.

$$\begin{array}{l} a^3+2a^2b+2ab^2+b^3 \\ a^3-2a^2b+2ab^2-b^3 \\ \hline a^6+2a^5b+2a^4b^2+a^3b^3 \\ \quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\ \qquad\quad +2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\ \qquad\qquad\qquad -a^3b^3-2a^2b^4-2ab^5-b^6 \\ \hline a^6 \quad \ast \quad \ast \quad \ast \quad \ast \quad \ast \quad -b^6 \end{array}$$

2d. Change the order of the question; that is, multiply the factors I. and III., then II. and IV. together.

Then multiply the products I. III., and the II. IV.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \quad \ast \quad -b^2 \end{array} \qquad \text{Then } \begin{array}{l} a^2+ab+b^2 \\ a^2-ab+b^2 \\ \hline a^4+a^3b+a^2b^2 \\ \quad -a^3b-a^2b^2-ab^3 \\ \qquad\quad +a^2b^2+ab^3+b^4 \\ \hline a^4 \quad \ast \quad +a^2b^2 \quad \ast \quad +b^4 \end{array} \qquad \begin{array}{l} a^4+a^2b^2+b^4 \\ x^2-b^2 \\ \hline a^6+a^4b^2+a^2b^4 \\ \quad -a^4b^2-a^2b^4-b^6 \\ \hline a^6 \quad \ast \quad \ast \quad -b^6 \end{array}$$

which is the product required.

3d. Again multiply the I. factor by the IV., and next the II. by the III.

It remains to multiply the product I. IV. and II. III

$$\begin{array}{l} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3 \quad \ast \quad \ast \quad +b^3 \end{array} \qquad \text{Next } \begin{array}{l} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 \\ \hline a^3 \quad \ast \quad \ast \quad -b^3 \end{array} \qquad \begin{array}{l} a^3+b^3 \\ a^3-b^3 \\ \hline a^6+a^3b^3 \\ \quad -a^3b^3-b^6 \\ \hline a^6 \quad \ast \quad -b^6 \end{array}$$

as in the two foregoing cases.

It will be proper to illustrate this example by a numerical application. Suppose $a=3$, and $b=2$, we shall have $a+b=5$, and $a-b=1$; further, $a^2=9$, $ab=6$, and $b^2=4$, therefore $a^2+ab+b^2=19$, and $a^2-ab+b^2=7$: so that the product required is that of

$5\times19\times1\times7=665$. Now $a^6=729$, and $b^6=64$; consequently the product is $a^6-b^6=665$, as we have already seen.

Ex. 4.

$$
\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc \\
a+b+c \\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc \\
\quad +a^2b+b^3+bc^2-ab^2-abc-b^2c \\
\qquad\qquad a^2c+b^2c+c^3-abc-ac^2-bc^2 \\
\hline
a^3 \;*\; +b^3 \;*\; -3abc+c^3
\end{array}
$$

Ex. 5.

$$
\begin{array}{r}
a^3+3a^2x+3ax^2+x^3 \\
a^3-3a^2x+3ax^2-x^3 \\
\hline
a^6+3a^5x+3a^4x^2+a^3x^3 \qquad\qquad\qquad\quad \\
-3a^5x-9a^4x^2-9a^3x^3-3a^2x^4 \qquad\quad \\
+3a^4x^2+9a^3x^3+9a^2x^4+3ax^5 \\
-a^3x^3-3a^2x^4-3ax^5-x^6 \\
\hline
a^6 \;*\; -3a^2x^4 \;*\; +3a^2x^4 \;*\; -a^6
\end{array}
$$

Ex. 6.

$$
\begin{array}{ll}
28-\surd(20x^2-10x) \text{ multiply} & \text{add } -2352\surd(20x^2-10x) \\
28-\surd(20x^2-10x) \text{ by} & \text{and } +(10x-20x^2)\surd(20x^2-10x) \\
\cline{1-1}\cline{2-2}
784-28\surd(20x^2-10x) & \text{sum } +(10x-20x^2-2352)\surd(20x^2-10x) \\
\quad -28\surd(20x^2-10x)+20x^2-10x & \\
\cline{1-1}
\multicolumn{2}{l}{784-56\surd(20x^2-10x)+20x^2-10x \text{ product. Multiply}} \\
\multicolumn{2}{l}{\;28-\surd(20x^2-10x) \text{ again by this.}} \\
\hline
\multicolumn{2}{l}{21952-1568\surd(20x^2-10x)+560x^2-280x} \\
\multicolumn{2}{r}{-784\surd(20x^2-10x)+56(20x^2-10x)-(20x^2+10x)\surd(20x^2-10x)} \\
\hline
\multicolumn{2}{l}{21952-2352\surd(20x^2-10x)+1680x^2-840x+(10x-20x^2)\surd(20x^2-10x)}
\end{array}
$$

Ex. 7.

$$
\begin{array}{l}
2352+20x^2-10x \text{ multiply} \\
2352+20x^2-10x \text{ by} \\
\hline
5531904+47040x^2-23520x \\
\qquad\qquad +47040x^2+400x^4-200x^3 \\
\qquad\qquad\qquad\qquad -23520x-200x^3+100x^2 \\
\hline
5531904-47040x+94180x^2-400x^3+400x^4 \text{ sum.} \\
\text{Multiply } 20x^2-10x \text{ again.} \\
\hline
110638080x^2-940800x^3+1883600x^4-8000x^5+8000x^6 \\
\qquad\qquad -55319040x+470400x^2-941800x^3+4000x^4-4000x^5 \\
\hline
111108480x^2-55319040x-1882600x^3+1887600x^4-12000x^5+8000x^6
\end{array}
$$

Ex. 8.

$7x^3-1676x^2+840x-21952$ multiply
$7x^3-1676x^2+840x-21952$ by

$$\begin{array}{l}
49x^6-11732x^5+5880x^4-153664x^3 \\
\qquad -11732x^5+2808976x^4-1407840x^3+36791552x^2 \\
\qquad\qquad +5880x^4-1407840x^3+705600x^2-18439680x \\
\qquad\qquad\qquad -153664x^3+36791552x^2-18439680x+481890304 \\
\hline
49x^6-23464x^5+2820736x^4-3123008x^3+74288704x^2-36879360x+481890304
\end{array}$$

NOTE. The products of the *powers* of the same quantity are found by adding their indices. Thus: $a^1\times a^1=a^2$, or $a\times a=a^{1+1}=a^2$; $a^1\times a^2=a^{1+2}=a^3$; $a^2\times a^3=a^5$; $a^m\times a^n=a^{m+n}$; $a^{\frac{1}{3}}\times a^{\frac{1}{4}}=a^{\frac{1}{3}+\frac{1}{4}}=a^{\frac{7}{12}}$; $a^{\frac{1}{m}}\times a^{\frac{1}{n}}=a^{\frac{m+n}{mn}}$; $x^n\times x=x^{n+1}$.

Obs. 1. In multiplication, as well as in addition and subtraction, the order of the letters is of no consequence. Thus, if *abc* be multiplied by *d*, the product is *abcd*, *bacd*, *cbda*, &c., each of which is of the same value; but it is usual to arrange them according to the order of the alphabet.

Obs. 2. In algebra it is customary to begin the multiplication on the left; but because the steps are merely indicated, it is of no consequence where the operation commences.

Obs. 3. It may be useful to observe, that, according to Euclid, Lib. II. Prop. V., the product of the sum and difference of any two quantities is equal to the difference of their squares; thus, (1.) $(a+b)(a-b)=a^2-b^2$
(2.) $(a^2+b^2)(a^2-b^2)=a^4-b^4=(a^2+b^2)(a+b)(a-b)$.
(3.) $(a^4+b^4)(a^4-b^4)=a^8-b^8=(a^4+b^4)(a^2+b^2)(a^2-b^2)=(a^4+b^4)(a^2+b^2)(a+b)(a-b)$
These compositions and decompositions of quantities are often found to be of great utility in the solution of equations.

Ex. 9.

Multiply $ex+mx^2+nx^3+rx^4$
by $ax+bx^2+cx^3+dx^4$

$$\begin{array}{l}
aex^2+amx^3+anx^4+arx^5 \\
\qquad bex^3+bmx^4+bnx^5+brx^6 \\
\qquad\qquad +cex^4+cmx^5+cnx^6+crx^7 \\
\qquad\qquad\qquad dex^5+dmx^6+dnx^7+drx^8 \\
\hline
aex^2+(am+be)x^3+(an+bm+ce)x^4+(ar+bn+cm+de)x^5 \\
\qquad +(br+cn+dm)x^6+(cr+dn)x^7+drx^8
\end{array}$$

Ex. 10.

$$\begin{array}{ll}
 & 114-\sqrt{\{196x^2-(x^2+24)^2\}}\text{ or, }114-\sqrt{(196x^2-x^4-48x^2-576)} \\
\text{or mult.} & 114-\sqrt{(148x^2-x^4-576)} \\
\text{by} & 114-\sqrt{(148x^2-x^4-576)} \\ \hline
 & 114^2-114\sqrt{(148x^2-x^4-576)} \\
 & \qquad -114\sqrt{(148x^2-x^4-576)}+(148x^2-x^4-576) \\ \hline
\text{Prod.} & 12996-228\sqrt{(148x^2-x^4-576)}+(148x^2-x^4-576) \\ \hline
\end{array}$$

Ex. 12.

$$\begin{array}{l}
3x^3+2x^2y^2+3y^3 \\
2x^3-3x^2y^2+5y^3 \\ \hline
6x^6+4x^5y^2+6x^3y^3 \\
\quad -9x^5y^2-6x^4y^4-9x^2y^5 \\
\qquad +15x^3y^3+10x^2y^5+15y^6 \\ \hline
6x^3-5x^5y^2-6x^4y^4+21x^3y^3+x^2y^5+15y^6 \\ \hline
\end{array}$$

Ex. 13.

$$\begin{array}{l}
a^4+a^2c^2+c^4 \\
a^2-c^2 \\ \hline
a^6+a^4c^2+a^2c^4 \\
\quad -a^4c^2-a^2c^4-c^6 \\ \hline
a^6 \quad * \quad * \quad -c^6
\end{array}$$

NOTE. When the quantities that are to be multiplied together have literal coefficients, proceed as before, putting the sum, or difference of the coefficients of the resulting terms, between brackets, as in the former rule. And if several compound quantities are to be multiplied together, multiply the first by the second, and then that product by the third, and so on to the last factor, as below.

$$\begin{array}{l}
ax+b \\
cx-d \\ \hline
acx^2+bcx \\
\quad -adx-bd \\ \hline
acx^2+(bc-ad)x-bd
\end{array}
\qquad\qquad
\begin{array}{l}
a-bx \\
c-dx \\ \hline
ac-bcx \\
\quad -adx+bdx^2 \\ \hline
ac-(bc+ad)x+bdx^2
\end{array}$$

$$\begin{array}{l}
a+2x \\
a-3x \\ \hline
a^2+2ax \\
\quad -3ax-6x^2 \\ \hline
a^2-ax-6x^2 \\
a+4x \\ \hline
a^3-a^2x-6ax^2 \\
\quad +4a^2x-4ax^2-24x^3 \\ \hline
a^3+3a^2x-10ax^2-24x^3 \\ \hline
\end{array}
\qquad\qquad
\begin{array}{l}
3a-x \\
2a+4x \\ \hline
6a^2-2ax \\
\quad +12ax-4x^2 \\ \hline
6a^2+10ax-4x^2 \\
4a-2x \\ \hline
24a^3+40a^2x-16ax^2 \\
\qquad -12a^2x-20ax^2+8x^3 \\ \hline
24a^3+28a^2x-36ax^2-8x^3 \\ \hline
\end{array}$$

To this we may add, that it is usual, in some cases, to write down the quantities that are to be multiplied together, between brackets, or under a vinculum, without performing the whole operation; as $3ab(a+b)\times a\sqrt{(a^2-b^2)}$

DIVISION.

41. Division in Algebra is the method of finding the quotient arising from the division of one algebraic quantity by another. Division is generally divided into three cases, namely, when the divisor and dividend are both simple quantities; when the divisor is a simple quantity and the dividend a compound one; and when the divisor and dividend are both compound quantities.

CASE I. *When the divisor and dividend are both simple terms.*

RULE. Place the divisor in the form of a denominator under the dividend; cancel those letters which are common to both, and divide the coefficients by any number that will divide them without a remainder, and the result will be the quotient required.

RULE II. Divide the coefficients as in common arithmetic, and to the quotient annex those letters in the dividend which are not found in the divisor.

A general rule for the signs in all the cases of division.

When the signs of the divisor and dividend are alike, (that is, both + or both —,) the sign of the quotient will be +. When they are unlike, (that is, the one + and the other —,) the sign of the quotient will be —.

The above rule briefly expressed in one view, is as follows:

	Div'r.	Div'd.	Quo't.
1.	+ Plus	+ Plus	+ Plus
2.	— Minus	— Minus	+ Plus
3	—	+	— Both minus.
	+	—	—

AGAIN, THUS:

$$\frac{+ab}{+a} = +b; \quad \frac{-ab}{-a} = +b; \quad \frac{-ab}{+a} = -b; \quad \frac{+ab}{-a} = -b;$$

and these four are all the cases that can possibly happen with regard to the variation of the signs.

Powers and roots of the same quantity are divided by subtracting their idices, that is, subtract the index of the divisor from the index of the dividend.

Thus, $\frac{a^2}{a} = a^{2-1} = a^1$, or a; $\frac{a^5}{a^3} = \frac{a^2 \times a^3}{a^3} = a^2 = a^{5-2}$;

$\frac{a^3}{a^5} = a^{3-5} = a^{-2}$, $\frac{a^{-5}}{a^{-2}} = a^{-5+2} = a^{-3}$; $\frac{a^{-2}}{a^{-5}} = a^{-2+5} = a^{+3}$; $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}} = a^{\frac{1}{2}-\frac{1}{2}} = a^o$; $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{1}{6}}$; $\frac{a^{\frac{3}{4}}}{a^{\frac{2}{5}}} = a^{\frac{3}{4}-\frac{2}{5}} = a^{\frac{7}{20}}$; $\frac{a}{a^n} = a^{m-n}$;

$\frac{a^{-m}}{a^n} = a^{-m-n}$; $\frac{a^{-m}}{a^{-n}} = a^{-m+n}$.

1. Divide $16x^2$ by $8x$.

$$\frac{16x^2}{8x} = 2x, \text{ Ans.}$$

2. Divide $12ax$ by $3a$.

$$\frac{12ax}{3a} = 4x, \text{ Ans.}$$

3. Divide $-10x^2$ by $-2x$.

$$\frac{-10x^2}{-2x} = +5x, \text{ Ans.}$$

4. Divide $-18ax^2y$ by $-8ax$.

$$\frac{-18ax^2y}{-8ax} = +\frac{9xy}{4}, \text{ Ans.}$$

5. Divide $12a^2x^2$ by $-8a^2x$.

$$\frac{12a^2x^2}{-8a^2x} = -\frac{3x}{2}, \text{Ans.}$$

6. Divide $-15ay^2$ by $3ay$.

$$\frac{-15ay^2}{3ay} = -5y, \text{ Ans.}$$

When the coefficients of the divisor and dividend are both fractions.

Rule. Invert the coefficients of the divisor, that is, put the denominator of the divisor in the numerator's place, and the numerator in the denominator's place; then proceed as in multiplication of fractions in arithmetic.

7. Divide $12a^2x^2$ *by* $3a^2x$.

$$\frac{12a^2x^2}{-3a^2x} = -4x, \text{ Ans.}$$

8. Divide $-15ab^{\frac{1}{2}}$ by $-3ab^{\frac{1}{2}}$.

$$\frac{-15ab^{\frac{1}{3}}}{-3ab^{\frac{1}{2}}} = +\frac{15ab^{\frac{2}{3}}}{3ab^{\frac{1}{2}}} = 5b^{\frac{2}{3}-\frac{1}{2}} = 5b^{\frac{1}{6}}.$$

9. Divide $18a^2b^2x^{\frac{1}{2}}$ by $-6ab^2x^{\frac{1}{4}}$

$$\frac{18a^2b^2x^{\frac{1}{2}}}{-6ab^2x^{\frac{1}{4}}} = -3ax^{\frac{1}{4}}.$$

Divide $-13a^{\frac{1}{2}}b^3c$ by $-7a^{\frac{1}{3}}b^2c^{\frac{1}{2}}$.

$$\frac{-13a^{\frac{1}{2}}b^3c}{-7a^{\frac{1}{3}}b^2c^{\frac{1}{4}}} = \frac{13a^{\frac{1}{2}-\frac{1}{3}}\ ac^{1-\frac{1}{2}}}{7} = \frac{13a^{\frac{1}{6}}bc^{\frac{1}{2}}}{7}$$

Case II. *When the divisor is a simple quantity and the dividend a compound one.*

Rule. Divide each term of the dividend separately by the divisor, and put down such as will not divide in the form of a fraction, and the result will be the quotient required.

1. Divide $18a^2b-54a^2$ by $6a^2$.

$$\frac{18a^2b-54a^2}{6a^2} = 3b-9.$$

2. Divide $16ax-40x^3$ by $8x$.

$$\frac{16ax-40x^3}{8x} = 2a-5x^2.$$

3. Divide $12a^2x+4ax^2-16a$ by $4a$.

$$\frac{12a^2x+4ax^2-16a}{4a} = 3ax+x^2-4, \text{ Ans.}$$

4. Divide $3x^3+6x^2+3ax-15x$, by $3x$. Ans., $x^2+2x+a-5$.

5. Divide $-8x^3+5x-12+2b$ by $-2x$.

$$\frac{-8x^3+5x-12+2b}{-2x} = +4x^2-\frac{5}{2}+\frac{6}{x}-\frac{b}{x}, \text{ Ans.}$$

6. Divide $a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x-$, &c., by a^n.

$$\frac{a^{n+1}x-a^{n+2}x-a^{n+3}x-a^{n+4}x}{a^n} = ax-a^2x-a^3x-a^4x-, \text{ Ans.}$$

CASE III. *When the divisor and dividend are both compound quantities.*

RULE. Arrange both divisor and dividend according to the powers of the same letters, beginning with the highest; then find how often the first term of the divisor is contained in the first term of the dividend, and place the result in the quotient; multiply the whole divisor by this quantity, and place the product under the corresponding or like terms in the dividend, and subtract it from them; to the remainder bring down as many terms of dividend as are requisite for the next operation, and divide as before, and so on till the work is finished, as in common arithmetic.

$$\begin{array}{l}
\text{Dividend.}\ \big\}\ 2x+3\ \text{Divisor.}\\
32x^5+243\ \big\}\ \overline{16x^4-24x^3+36x^2-54x+81}\ \text{quotient.}\\
\underline{32x^5+48x^4}\\
\quad -48x^4+243\\
\quad \underline{-48x^4-72x^3}\\
\qquad\quad 72x^3+243\\
\qquad\quad \underline{72x^3+108x^2}\\
\qquad\qquad -108x^2+243\\
\qquad\qquad \underline{-108x^2-162x}\\
\qquad\qquad\qquad 162x+243\\
\qquad\qquad\qquad \underline{162x+243}
\end{array}$$

$$\begin{array}{l}
\text{Divis.}\quad \text{Divid.}\qquad \text{Quotient.}\\
3x-6)6x^4-96(2x^3+4x^2+8x+16\\
\qquad\ \underline{6x^4-12x^3}\\
\qquad\qquad 12x^3-96\\
\qquad\qquad \underline{12x^3-24x^2}\\
\qquad\qquad\quad 24x^2-96\\
\qquad\qquad\quad \underline{24x^2-48x}\\
\qquad\qquad\qquad 48x-96\\
\qquad\qquad\qquad \underline{48x-96}
\end{array}$$

$$\begin{array}{l}
\qquad\text{Dividend.}\qquad\qquad\quad \big\}\ 2x-3a\ \text{Divisor.}\\
48x^3-76ax^2-64a^2x+105a^3\ \big\}\ \overline{24x^2-2ax-35a^2}\ \text{quot.}\\
\underline{48x^3-72ax^2}\\
\quad -4ax^2-64a^2x\\
\quad \underline{-4ax^2+6a^2x}\\
\qquad\quad -70a^2x+105a^3\\
\qquad\quad \underline{-70a^2x+105a^3}
\end{array}$$

$$\begin{array}{l}
\qquad\text{Dividend.}\qquad\qquad \text{Divisor.}\\
4x^4-9x^2+6x-1\ \big\}\ 2x^2+3x-1\\
\underline{4x^4+6x^3-2x^2}\ \big\}\ \overline{2x^2-3x+1}\ \text{quot.}\\
\quad -6x^3-7x^2+6x\\
\quad \underline{-6x^3-9x^2+3x}\\
\qquad\quad +2x^2+3x-1\\
\qquad\quad\ \ 2x^2+3x-1
\end{array}$$

$$\begin{array}{l}
\text{Dividend.}\\
x^4+y^4\ \big\}\ x+y\ \text{divisor.}\\
\underline{x^4+x^3y}\ \big\}\ \overline{x^3-x^2y+xy^2-y^3+\dfrac{2y^4}{x+y}}\ \text{quot.}\\
\ -x^3y+y^4\\
\ \underline{-x^3y-x^2y^2}\\
\qquad +x^2y^2+y^4\\
\qquad\ \ \underline{x^2y^2+xy^3}\\
\qquad\qquad -xy^3+y^4\\
\qquad\qquad \underline{-xy^3-y^4}\\
\qquad\quad \text{Remainder } 2y^4
\end{array}$$

$$\begin{array}{l}
\text{Dividend.}\ x^3-y^3\ \big\{\ x+y\ \text{Divisor.}\\
\qquad\qquad \underline{x^3+x^2y}\ \big\{\ \overline{x^2-xy+y^2-\dfrac{2y^3}{x+y}}\ \text{quot.}\\
\qquad\qquad -x^2y-y^3\\
\qquad\qquad \underline{-x^2y-xy^2}\\
\qquad\qquad\qquad xy^2-y^3\\
\qquad\qquad\qquad \underline{xy^2+y^3}\\
\qquad\qquad \text{Remainder}-2y^3
\end{array}$$

1. Divide $x^3-x^2-13x-11$ by $x+1$. Ans. $x^2--2x-11$.

Answer $x^3-x^2-13x-11$, and $x^2-2x-11$.

$-y)x^2+2xy+y^2(x+y$
x^2+xy
$xy+y^2$
$xy+y^2$

dividend. $a+x$ divisor.
$a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2$
a^3+a^2x quotient.
$4a^2x+5ax^2$
$4a^2x+4ax^2$
ax^2+x^3
ax^2+x^3

Dividend.
$-3)x^3-9x^2+27x-27(x^2-6x+9$ quotient.
x^3-3x^2
$-6x^2+27x$
$-6x^2+18x$
$9x-27$
$9x-27$

Divisor. Dividend.
$y-8)2y^3-19y^2+26y-16(2y^2-3y+2$
$2y^3-16y^2$ quotient.
$-3y^2+26y$
$-3y^2+24y$
$2y-16$
$2y-16$

$-y)b^4-3y^4(b^3+b^2y+by^2+y^3-\frac{2y^4}{b-y}$ quotient.
b^4-b^3[illegible]
b^3y-3y^4
$b^3y-b^2y^2$
$b^2y^2-3y^4$
$b^2y^2-by^3$
by^3-3y^4
by^3-y^4
$-2y^4$ remainder.

$a+x)a^3-x^3(a^2-ax+x^2-\frac{2x^2}{a+x}$
a^3+a^2x quotient.
$-a^2x-x^3$
$-a^2x-ax^2$
ax^2-x^3
ax^2+x^3
$-2x^3$

vis. Divid.
$-x)a^3+5a^2x+5ax^2+x^3(a^2+6ax+11x^2+\frac{12x^3}{a-x}$ quotient.
a^3-a^2x
$6a^2x-6ax^2$
$6a^2x+5ax^2$
$11ax^2+x^3$
$11ax^2-11x^3$
Remainder $12x^3$

Divis. Divid.
$x-a)x^3-3ax^2+3a^2x-a^3(x^2-2ax+a^2x$
x^3-ax^2
$-2ax^2+3a^2x$
$-2ax^2+2a^2x$
a^2x-a^3
a^2x-a^3

Divisor. Dividend. Quot.
$-2ax+x^2)a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5(a^3-3a^2x$
$a^5-2a^4x+a^3x^2$ $+3ax^2-x^3$
$-3a^4x+9a^3x^2-10a^2x^3$
$-3a^4x+6a^3x^2-3a^2x^3$
$3a^3x^2-7a^2x^3+5ax^4$
$3a^3x^2-6a^2x^3+3ax^4$
$-a^2x^3+2ax^4-x^5$
$-a^2x^3+2ax^4-x^5$

ON THE REDUCTION OF ALGEBRAIC FRACTIONS.

(42.) The rules managing algebraic fractions being of the same nature as vulgar fractions in common arithmetic, the operations are performed exactly in the same manner.

Case I. *To reduce a mixed quantity to an improper fraction.*

Rule. Multiply the integer, or whole part, by the denominator of the fraction, and to the product add the numerator; then under their sum place the original denominator.

1. Reduce $a+\frac{b}{a}$ and $a-\frac{x}{b}$ each to improper fractions.

$a+\frac{b}{a}=\frac{a\times a+b}{a}=\frac{a^2+b}{a}$ Ans. $a-\frac{x}{b}=\frac{ab-x}{b}$ Ans.

2. Reduce $4x-\frac{3x-5}{4}$ and $4ax-\frac{b^2}{2ax}$ to improper fractions.

Here $4x-\frac{3x-5}{4}=\frac{16x-3x+5}{4}=\frac{13x+5}{4}$ Ans.

and $4ax-\frac{b^2}{2ax}=\frac{4ax\times 2ax-b}{2ax}=\frac{8a^2x^2-b^2}{2ax}$ Ans.

3. Reduce $x-\frac{a^2-x^2-c}{x}$ and $y-\frac{b^2-y^2-d}{y}$ to improper fractions.

$x-\frac{a^2-x^2-c}{x}=\frac{x^2-a^2+x^2+c}{x}=\frac{2x^2-a^2+c}{x}$ Ans. $\frac{2y^2-b^2+d}{y}$

Case II. To reduce an improper fraction to a whole, or mixed quantity. This is the same as Case II. in division.

Rule. Divide the numerator by the nominator for the integral part; and if there be a remainder, place it over the denominator for the fractional part, with the proper sign prefixed.

1. Reduce $\frac{ab-2a^2}{ab}$, $\frac{a^2+x^2}{a\pm x}$, and $\frac{10x^2-5x+3}{5x}$ to whole quantities.

$\frac{ab-2a^2}{ab}=1-\frac{2a}{b}$ Ans. $\frac{10x^2-5x+3}{5x}=2x-1+\frac{3}{5x}$ Ans.

and $a\pm a)a^2+x^2(a\pm x+\frac{2x^2}{a\pm x}$ Answer as required.

$\underline{a^2\pm ax}$

$\mp ax+x^2$ Here $a\pm x$ is the integral part of the quantity.

$\underline{\pm ax-x^2}$

Remainder $2x^2$ and $\frac{2x^2}{a\pm x}$ is the fractional part. Therefore $a\pm x+\frac{2x^2}{a\pm x}$ is the mixed quantity required.

2. Reduce $\frac{x^3+y^3}{x+y}$, $\frac{y^3-y^3}{x-y}$, and $\frac{x^2+4ax+4a^2-y^2}{x+2a}$ to their proper terms. Ans. $x^2 \pm xy+y^2$, and $x+2a-\frac{y^2}{x+2a}$ Answer.

Case III. *Fractions reduced to a common denominator.*

Rule. Multiply each numerator separately by every denominator except its own, for the new numerator, and all the denominators together for a common denominator.

1. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$; or $\frac{ax}{3}$ and $\frac{x}{4}$. Also, $\frac{x}{y}$ and $\frac{x+y}{z}$ each pair to equivalent fractions, that shall have common denominators.

$$\left.\begin{array}{l} 2x\times c=2cx \\ a\times b=ab \end{array}\right\}\text{ the numerators} \qquad \left\{\begin{array}{l}\text{The new fractions are} \\ \frac{2cx}{ac} \text{ and } \frac{ab}{ac}\text{ Ans.}\end{array}\right.$$
$$a\times b=ac \text{ the common denominator}$$

$$\left.\begin{array}{l} ax\times 4=4ax \\ 3\times x=3x \end{array}\right\}\text{ new numerators.} \qquad \left\{\begin{array}{l}\text{The new fractions are} \\ \frac{4ax}{12} \text{ and } \frac{3x}{4}\end{array}\right.$$
$$3\times 4=12 \text{ common denominator}$$

$$\left.\begin{array}{l} x\times z=xz \\ (x+y)\times y=xy+y^2 \end{array}\right\}\text{ new numerator.} \qquad \left\{\begin{array}{l}\text{The new fractions are} \\ \frac{xy}{yz} \text{ and } \frac{xy+y^2}{yz}\end{array}\right.$$
$$y\times z=yz \text{ common denominator.}$$

2. Reduce $\frac{2x+3}{x}$ and $\frac{5x+1}{3}$; again, $\frac{7x^2-1}{2x}$ and $\frac{4x^2-x+2}{2a^2}$ each pair of fractions to a common denominator.

Ans. $\frac{6x+9}{3x}$ and $\frac{5x^2+x}{3x}$; Ans. $\frac{14a^2x^2-2a^2}{4a^2x}$, and $\frac{8x^3-2x^2+4x}{4a^2x}$

3. Reduce $\left.\frac{a+x}{a-x} \text{ and } \frac{a-x}{a+x}\right\}$ or $\frac{5x}{a+x}$, $\frac{a-x}{3}$ and $\frac{1}{2x}$ to a common denominator.

$$\left.\begin{array}{l} (a+x)\times(a+x)=a^2+2ax+x^2 \\ (a-x)\times(a-x)=a^2-2ax+x^2 \end{array}\right\}\text{ new numerators.}$$
$$\left.\begin{array}{l} (a-x)\times(a+x)=a^2-x^2 \\ \text{The common denominator.} \end{array}\right\}\text{ Ans.}\frac{a^2+2ax+x^2}{a^2-x^2} \text{ and } \frac{a^2-2ax+x^2}{a^2-x^2}$$

$$\left.\begin{array}{l} \text{Here } 5x\times 3\times 2x=30x^2 \\ (a-x)\times(a+x)2x=2a^2x-2x^3 \\ 1\times(a+x)\times 3=3a+3x \\ \hline (a+x)\times 3\times 2x=6ax+6x^2 \end{array}\right\} \begin{array}{c} \therefore\text{ The new fractions are} \\ \frac{30x^2}{6ax+6x^2},\quad \frac{2a^2x-2x^3}{6ax+6x^2} \text{ and} \\ \frac{3a+3x}{6ax+6x^2} \end{array}$$

4. Reduce $\frac{a}{3}$, $\frac{2y}{b}$, and $\frac{a+y}{a-y}$; or $\frac{x}{3}$, $\frac{x+1}{5}$ and $\frac{1-x}{1+x}$ to a common denominator.

Here
$$\left.\begin{array}{l} x\times5\times(1+x)=5x^2+5x \\ (x+1)\times3\times(1+x)=3x^2+6x+3 \\ (1-x)\times3\times5=15-15x \\ \hline 3\times5\times(1+x)=15+15x \text{ common denominator.} \end{array}\right\} \therefore \text{ the new fractions are } \frac{5x^2+5x}{15+15x}, \quad \frac{15-15x}{15+15x} \text{ and } \frac{3x^2+6x+3}{15+15x} \text{ Ans.}$$

$$\left.\begin{array}{l} a\times b\times(a-y)=a^2b-aby \\ 2y\times3\times(a-y)=6ay-6y^2 \\ (a+y)\times3\times b=3ab+3by \\ \hline 3\times b\times(a-y)=3ab-3by \\ \quad\text{common denominator.} \end{array}\right\} \therefore \text{ the new fractions are } \frac{a^2b-aby}{3ab-3by}, \quad \frac{6ay-6y^2}{3ab-3by} \text{ and } \frac{3ab+3by}{3ab-3by} \text{ Ans.}$$

5. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a+x}{a-x}$: Again, $\frac{3}{4}$, $\frac{3x}{3}$ and $a+\frac{4x}{5}$ each set of fractions to a common denominator.

$$\left.\begin{array}{l} a\times7\times(a-x)=7a^2-7ax \\ 3x\times2\times(a-x)=6ax-6x^2 \\ (a+x)\times2\times7=14a+14x \end{array}\right\} \text{ new numerators.}$$

$2\times7\times(a-x)=14a-14x$ common denominator.

CASE IV. *To find the greatest common measure, or divisor, of the terms of a fraction, consisting of compound quantities, or to reduce a fraction to its lowest, or most simple terms.*

RULE I. Range the quantities according to the powers, or dimensions of some letter, as in Division, and by inspection expunge the common factor or factors, if any, either in the divisor or dividend; then divide that quantity which has the highest power by the other, whether it be the numerator or denominator; and divide the last divisor by the last remainder, and so on till nothing remains; the last divisor will be the greatest common measure: but if such a divisor cannot be found, the fraction has no common measure. Having found the greatest common measure, divide the terms of the fraction by it, and the resulting fraction will be in its lowest terms.

NOTE. If any of the divisors become negative, they may have their signs changed without altering the truth of the result. Also, if the first term of a divisor be not contained an exact number of times in the first term of the dividend, the latter may be multiplied by any quantity that will make the division complete. And lastly, any quantity which is common to all the terms of the dividend or divisor, may be expunged before the division is commenced.

1. Find the greatest common measure of $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ and then reduce the fraction to its lowest terms.

$$\begin{array}{l} 6a^2+7ax-3x^2)6a^2+11ax+3x^2(1 \\ \qquad\qquad\qquad\;\; \underline{6a^2+7ax-3x^2} \end{array}$$

Now dividing this $4ax+6x^2$ remainder by $2x$, the quotient is $2a+3x$; and dividing the divisor $6a^2+7ax-3x^2$ by this quotient, we have $2a+3x)6a^2+7ax-3x^2(3a-x$

$6a^2+9ax$

$-2ax-3x^2$

$-2ax-3x^2$, and since there is no remainder, therefore $2a+3x$, the last divisor, is the greatest common measure required. $2a+3x)\dfrac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}=\dfrac{3a-x}{3a+x}$ Ans.

In this example the dimension of the numerator is the same as that of the denominator; therefore, it makes no difference whether we divide the numerator by the denominator, or the denominator by the numerator. This will be evident from the following operation: $6a^2+11ax+3x^2)6a^2+7ax-3x^2(1$

$6a^2+11ax+3x^2$

Now, dividing this $-4ax-6x^2$ remainder by $-2x$, the quotient is $2a+3x$; or, as the remainder above is now, negative. And, by changing all the signs, we obtained $4ax+6x^2$, or $2a+3x$, for the new divisor. $2a+3x)6a^2+11ax+3x^2(3a+x$

$6a^2+9ax$

$2ax+3x^2$

$2ax+3x^2$ Hence $2a+3x$ is found to be the greatest common measure, as before.

2. Required to find the greatest common measure of $\dfrac{x^2-xy-2y^2}{x^2-3xy+2y^2}$ or $\dfrac{x^2-1}{ax+a}$ or $\dfrac{7x^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$ and $\dfrac{x^3-a^3}{x^4-a^4}$ and then reduce each fraction to their lowest terms.

$x^2-xy-2y^2)x^2-3xy+2y^2(1$

$x^2-xy-2y^2$

Divide this $-2xy+4y^2$ remainder, each term of which can be divided by $2y$, and the signs of the terms being changed, we obtain $x-2y$ for a new divisor. $x-2y)x^2-xy-2y^2(x+y$

x^2-2xy

$+xy-2y^2$

$+xy-2y^2$ and since there is no remainder, therefore $x-2y$, the last divisor, is the greatest common measure required.

$$x-2y)\frac{x^2-xy-2y^2}{x^2-2xy+3y^2}=\frac{x+y}{x-y}\text{ Answer.}$$

In this example, the dimensions of the numerator is the same as that of the denominator; therefore it makes no difference whether we divide the numerator by the denominator, or the denominator by the numerator. The same remark will apply to the next example. This will be evident from the following operation.

$$x^2-3xy+2y^2)x^2-xy-2y^2(1$$
$$\underline{x^2-3xy+2y^2}$$

$2xy-4y^2$ the remainder, which is now positive, and divide by $+2y$, and we obtain $x-2y$, for our new divisor, the same as before.

$$\left.\begin{array}{l} ax+a, \text{ or} \\ x+1 \end{array}\right\} \begin{array}{l} x^2-1(x-1, \text{ the least common measure.} \\ \underline{x^2+x} \\ -x-1 \\ -x-1 \end{array}$$

$$x+1)\frac{x^2-1}{ax+a}=\frac{x-1}{a} \text{ Ans.}$$

3d expression solved. Multiply the denominator by 7, we shall have $7a^2-23ab+6b^2)35a^3-126a^2b+77ab^2-42b^3(5a-11b$

$$\underline{35a^3-115a^2b+30ab^2}$$
$$-11a^2b+47ab^2\ -42b^3$$
$$\text{Multiply by } 7$$
$$\overline{-77a^2b+329ab^2-294b^3}$$
$$\underline{-77a^2b+253ab^2-66b^3}$$

Divide this remainder by $76b^2$ and $76ab^2-228b^3$ we have for the divisor $a-3b)7a^2-23ab+6b^2(7a-2b$

$$\underline{7a^2-21ab}$$
$$-2ab+6b^2$$
$$-2ab+6b^2$$

Therefore $a-3b$ is the greatest common measure sought.

$$a-3b)\ \frac{7x^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}=\frac{7a-2b}{5a^2-3ab+2b^2} \text{Ans.}$$

Now, for the last fraction in example second, the greatest common measure of the fraction is $x-a)\dfrac{x^3-a^3}{x^4-a^4}=\dfrac{x^2+ax+a^2}{x^3+ax^2+a^2x+a^3}$ Ans.

ADDITION OF FRACTIONS IN ALGEBRA.

Rule. Reduce the fractions to a common denominator, add the numerators together, and under this sum place the common denominator; the result will be the sum of the fractions required.

1. Add $\dfrac{3x}{2a}$ and $\dfrac{x}{5}$; or $\dfrac{x}{2}$, $\dfrac{x}{3}$ and $\dfrac{x}{4}$; or $\dfrac{4x}{7}$ and $\dfrac{x-2}{5}$ together.

$$\frac{3x\times5+2a\times x}{2a\times5=10a}=\frac{15x+2ax}{10a}, \text{ or } \frac{3\times4\times x}{2\times3\times4}+\frac{2\times x\times4}{2.3.4}+\frac{x\times2\times3}{2.3.4}$$
$$=\frac{13x}{12};\ \frac{4x\times5+7(x-2)}{5\times7}=\frac{20x+7x-14}{35}=\frac{27x-14}{35}.$$

2. Add $2a$, $3a+\dfrac{2x}{5}$ and $a-\dfrac{8x}{9}$; or $2a+\dfrac{3x}{5}$, $\dfrac{a}{a-x}$ and $\dfrac{a-x}{a}$ together.

$$2a+3a+a=6a, +\frac{2x}{5}-\frac{8x}{9}=\frac{18x-40x}{5\times9}=6a-\frac{22x}{45}.$$

3. Add $\frac{2x}{5}$ and $\frac{5x}{7}$, or $15x$ and $\frac{1+2x}{8}$ or $\frac{ax}{b-c}$ and $\frac{ax}{b+c}$ together.

$$\frac{2x.7+5.5x}{5\cdot7}=\frac{14x+25x}{35}=\frac{39x}{35}, \text{ or } \frac{15x.8+1+2x}{1\times8}=\frac{120x+1+2x}{8}$$
$$=\frac{122x+1}{8}, \quad \frac{ax(b+c)+ax(b-c)}{(b-c)\times(b+c)}=\frac{abx+acx+abx-acx}{b^2-c^2=(b+c)\ (b-c)}=$$
$$\frac{2abx}{b^2-c^2}, \text{ Ans.}$$

4. Add $\frac{12x}{7}$ and $\frac{3x}{5}$ or $\frac{5x}{3}$ and $\frac{3y}{8}$ or $\frac{10x-9}{8}$ and $\frac{3x-5}{7}$ together.

$$\frac{12x\cdot5+7\cdot3x}{7\cdot5}=\frac{60x+21x}{35}=\frac{81x}{35}, \text{ or } \frac{5y\cdot8+3\cdot3y}{3\cdot8}=\frac{40y+9y}{24}=$$
$$\frac{49y}{24}, \text{ or } \frac{(10x-9)\times7+(3x-5)\times8}{8\cdot7=8\times7=56}=\frac{70x-63+24x-40}{56}=$$
$$\frac{94x-103}{56}, \text{ Ans.}$$

5. Add $\frac{a}{b}$, $\frac{2a}{3b}$ and $\frac{5b}{4a}$ or $\frac{2x+3}{5}$, $\frac{3x-1}{2x}$ and $\frac{4x}{7}$ together.

$$\left.\begin{array}{r} a\times3b\times4a=12a^2b \\ 2a\times\ b\times4a=\ 8a^2b \\ 5b\times\ b\times3b=15b^3 \end{array}\right\} \therefore\frac{12a^2b+8a^2b+15b^3}{12ab^2}=\frac{20a^2b+15b^3}{12ab^2}=(\text{di-}$$
$$b\times3b\times4a=12ab^2 \quad \text{viding by } b)\ \frac{20a^2+b^3}{12ab} \text{ Ans.}$$

$$\left.\begin{array}{r} (2x+3)\times2x\times7=28x^2+42x \\ (3x-1)\times5\ \times7=105x-35 \\ 4x\times5\ \times2x=\ 40x^2 \\ \hline 5\times2x\times7=70x=\text{com. den.} \end{array}\right\} \begin{array}{l} \therefore\dfrac{28x^2+42x+105x-35+40x^2}{70x} \\ =\dfrac{68x^2+147x-35}{70x} \text{ Ans.} \end{array}$$

SUBTRACTION OF ALGEBRAIC FRACTIONS.

51. **Rule.** Reduce the fractions to a common denominator, if required; then place the difference of their numerators over the common denominator, and it will be the difference required.

1. From $\frac{4x}{5}$ take $\frac{3x+1}{x+1}$, or $\frac{4x+2}{3}$ take $\frac{2x-3}{3x}$, or $\frac{1}{x-y}$ take $\frac{1}{x+y}$.

$$\frac{4x}{5}-\frac{3x+1}{x+1}=\frac{4x(x+1)-5(3x+1)}{5(x+1)}=\frac{4x^2+4x-15x-5}{5x+5}=$$
$$\frac{4x^2-11x-5}{5x+5}, \text{ or } \frac{(4x+2)3x-(2x-3)3}{3\times3x=9x}=\frac{12x^2+6x-6x+9}{9x}=$$

$\frac{12x^2+9}{9x}=\frac{4x^2+3}{3x}$, or $\frac{1}{x-y}-\frac{1}{x+y}=\frac{1\times(x+y)-1\times(x-y)}{(x-y)(x+y)=x^2-y^2}$
$=\frac{2y}{x^2-y^2}$, Ans. to the last.

2. From $a+\frac{a-x}{a(a+x)}$ take $\frac{a+x}{a(a-x)}$, or $a+\frac{a-x}{a+x}$ take $a-\frac{a+x}{a-x}$.
Reduced $\frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$ are $\frac{a^2-2ax+x^2}{a(a^2-x^2)}$, and $\frac{a^2+2ax+x^2}{a(a^2-x^2)}$ $\therefore$
$a+\frac{a^2-2ax+x^2}{a(a^2-x^2)}-\frac{a^2+2ax+x^2}{a(a^2-x^2)}=a-\frac{4x}{a(a^2-x^2)}$, Ans.
2d. Reduced are $a+\frac{a^2-2ax+x^2}{a^2-x^2}-(a-\frac{a^2+2ax+x^2}{a^2-x^2})=\frac{2a^2+2x^2}{a^2-x^2}$.

MULTIPLICATION OF ALGEBRAIC FRACTIONS.

52. Rule. Multiply all the numerators together for the numerator of the product, and all the denominators together for its denominator; the former placed over the latter will give the product required.

1. Multiply $\frac{2x}{5}$ by $\frac{3x}{2}$, or $\frac{3x}{2}$ by $\frac{5x}{3b}$, or $\frac{2x}{x-1}$ by $\frac{3x}{7}$.

$\frac{2x\times3x}{5\times2a}=\frac{6x^2}{10a}=\frac{3x^2}{5a}$, or $\frac{5x\times3x}{2\times3b}=\frac{15x^2}{2b}$, or $\frac{2x\times3x}{7(x-1)}=\frac{6x^2}{7x-7}$.

2. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$, or $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$.

$\frac{3ax-5a}{4x^2-6}$, or $\frac{3x^2(15x-30)}{2x(5x-10)}=\frac{3x^2\times3}{2x}=\frac{9x^2}{2x}=\frac{9x}{2}$, Ans.

3. Multiply $\frac{x-2}{5}$ by $\frac{4x}{7}$, or $2x$, $\frac{3x}{4}$ by $\frac{5x}{7}$, or $7x$, $\frac{2x}{5}$ by $\frac{3x^2}{2a}$.

$\frac{(x-2)4x}{5\times7}=\frac{4x^2-8x}{35}$, or $\frac{2x\times3x\times5x}{1\times4\times7}=\frac{30x^3}{28}=\frac{15x^3}{14}$, Ans.

Lastly, $\frac{7x\times2x\times3x^2}{1\times5\times2a}=\frac{7x\times x\times3x^2}{1\times5\times a}=\frac{21x^4}{5a}$, Ans.

8. Multiply $a+\frac{x}{2a}-\frac{x^2}{4a^2}$ by $x-\frac{a}{2x}+\frac{a^2}{4x^2}$.

By reduction I have $(\frac{4a^3+2ax-x^2}{4a^2})\times(\frac{4x^3-2ax+a^2}{4x^2})=$
$\frac{4a^5-8a^4x+16a^3x^3-5a^2x^2+2a^3x+2ax^3+8ax^4-4x^5}{16a^2x^2}$, Ans.

DIVISION OF FRACTIONS IN ALGEBRA.

53. **Rule.** Invert the divisor, and then proceed as in Multiplication.

EXAMPLES.

1. Divide $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$ or $\frac{4x^2}{7}$ by $5x$, or $\frac{9y^2-3y}{7}$ by $\frac{3y^2}{7}$, or $\frac{x+1}{6}$ by $\frac{2x}{3}$.

$\frac{x^2-9}{5} \div \frac{x+3}{4} = \frac{(x^2-9)4}{(x+3)5} = \frac{4x-12}{5}$ Ans., or $\frac{4x^2}{7} \div 5 = \frac{4x^2}{7} \times \frac{1}{5x} = \frac{4x^2}{35}$ Ans. $\frac{9y^2-3y}{7} \div \frac{3y^2}{7} = \frac{9y^2-3y}{7} \times \frac{7}{3y^2} = \frac{3y-1}{y}$ Ans., or $\frac{x+1}{6} \div \frac{2x}{3} = \frac{3x+3}{6\times 2x} = \frac{x+1}{4x}$.

3. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$ or $\frac{x^4-b^4}{x^2-2b+b^2}$ by $\frac{x^2+bx}{x-b}$.

$\frac{(2a+x)x}{c^3-x^3} \div \frac{x}{c-x} = \frac{2ax+x^2}{c^3-x^3} \times \frac{c-x}{x} = \frac{2a+x}{c^2+cx+x^2}$, Ans.

Again, $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b} = \frac{x^4-b^4}{(x-b)^2} \times \frac{x-b}{x^2+bx} = \frac{(x^2+b^2)\ (x^2-b^2)\ (x-b)}{(x-b)\ (x-b)\ (x+b)\times x} = \frac{(x^2+b^2)\ (x^2-b^2)}{(x^2-b^2)\times x} = \frac{x^2+b^2}{x}$, Ans.

4. Divide $\frac{x^2+xy}{x-y}$ by $\frac{x^4-y^4}{(x-y)^2}$ or $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

$\frac{(x^2+xy)\ (x^2-2xy+y^2)}{(x-y)\ (x^4-y^4)} = \frac{(x^2+xy)\ (x-y)}{x^4-y^4} = \frac{x^2+xy=x(x+y)}{x^3+x^2y+xy^2+y^3}$.

$\frac{5a^4 - 5b^4}{2a^2 - 4ab + 2b^2} \div \frac{6a^2 + 5ab}{4a - 4b} = \frac{(5a^4 - 5b^4) \times (4a - 4b)}{(2a^2-4ab-2b^2)\ (6a^2+5ab}$.

$\frac{(5\times 2)\ (a^4-b^4)}{(a-b)\ (6a^2+5ab)} = \frac{5.2.(a^3+a^2b+ab^2+b^3}{1\times(6a^2+5ab)} = \frac{10(a^3+a^2b+ab^2+b^3}{6a^2+5ab}$.

INVOLUTION.

54. Involution is the raising of any given quantity to any proposed power; such as the square, cube, &c.

If a quantity be continually multiplied by itself, it is said to be involved, or raised; and the power to which it is raised is expressed by the number of times the quantity has been employed in the multiplication; thus $a \times a$, or a^2, is called the second power of a; $a \times a \times a$, or a^3, the cube, or third power; $a \times a \ldots \surd(n)$, or a^n, the nth power.

Rule. When the quantity or root has no index, its power will be represented by placing the index of the required power above it; thus the fourth power of x is x^4; the fourth power of $x+y$ is $(x+y)^4$. If the quantity proposed be a compound one, the involution may either be represented by the proper index, or it may actually take place. If the quantity to be involved be negative, the signs of the even powers will be positive, and the signs of the odd powers negative; for $-a \times -a = a^2$; $-a \times -a \times -a = -a^3$.

If the quantity be a fraction, raise both the numerator and denominator to the same power. When the quantity to be raised is itself a power, multiply the index of the quantity by the index of the power; thus the cube of a^2 is $a^6 = a^{2 \times 3}$: or generally let m or n represent any powers whatever; then x^m, raised to the nth power is $x^{m \times n}$, or x^{mn}; and $-x^m$, raised to any power, n, will give $+$, plus, or $-$, minus, according as n is an odd or even number. If n be used for an uneven number, the sign will be $+$, if odd $-$; then $-x^m$ to the nth power, will be represented by $\pm x^{mn} = \pm x^{m \times n}$.

Roots and Powers of Numbers.

Root	1	2	3	4	5	6	7	8	9	10
Square	1	4	9	16	25	36	49	64	81	100
Cube	1	8	27	64	125	216	343	512	729	1000
4th power	1	16	81	256	625	1296	2401	4096	6561	10000
5th power	1	32	243	1024	3125	7776	16807	32768	59049	100000

The operation is performed in the same manner for simple algebraic quantities, except that in this case it must be observed, that the powers of negative quantities are alternately $+$ and $-$; the even powers being positive, and the odd powers negative. Thus the square of $+2a$ is $+2a \times +2a$, or $4a^2$; the square of $-2a$ is $-2a \times -2a$, or $+4a^2$; but the cube of $-2a = -2a \times -2a \times -2a = +4a^2 \times -2a = -8a^3$.

The several powers of $\frac{a}{b}$ are, square $= \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$, cube $= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$, 4th power $= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^4}{b^4}$.

Upon this principle the powers of the several roots in the following table are calculated.

Roots and Powers of Simple Algebraic Quantities.

Root	a	$-b$	$2b^2$	$\frac{a}{2b}$	$-\frac{3x^2}{y}$	$\frac{2a}{3b}$	a^2b	$-\frac{a^2}{b^3}$	$-\frac{3x}{5}$	$\frac{x}{4y}$
Square	a^2	$+b^2$	$4b^4$	$\frac{a^2}{4b^2}$	$+\frac{9x^4}{y^2}$	$\frac{4a^2}{9b^2}$	a^4b^2	$+\frac{a^4}{b^6}$	$+\frac{9x^2}{25}$	$\frac{x^2}{16y^2}$
Cube	a^3	$-b^3$	$8b^6$	$\frac{a^3}{8b^3}$	$-\frac{27x^6}{y^3}$	$\frac{8a^3}{27b^3}$	a^6b^3	$-\frac{a^6}{b^9}$	$-\frac{27x^3}{125}$	$\frac{x^3}{64y^3}$
4th power	a^4	$+b^4$	$16b^8$	$\frac{a^4}{16b^4}$	$+\frac{81x^8}{y^4}$	$\frac{16a^4}{81b^4}$	a^8b^4	$+\frac{a^8}{b^{12}}$	$+\frac{81x^4}{625}$	$\frac{x^4}{256y^4}$
5th power	a^5	$-b^5$	$32b^{10}$	$\frac{a^5}{32b^5}$	$-\frac{243x^{10}}{y^5}$	$\frac{32a^5}{243b^5}$	$a^{10}b^5$	$-\frac{a^{10}}{b^{15}}$	$-\frac{243x^5}{3125}$	$\frac{x^5}{1024y^5}$

What is the square of $a+2b$?

$$\begin{array}{rl}
 & a+2b \\
 & a+2b \\ \hline
 & a^2+2ab \\
 & \quad +2ab+4b^2 \\ \hline
\text{Sq.} = & a^2+4ab+4b^2 \\
 & a+2b \\ \hline
 & a^3+4a^2b+4ab^2 \\
 & \quad +2a^2b+8ab^2+8b^3 \\ \hline
\text{Cube} & a^3+6a^2b+12ab^2+8b^3
\end{array}$$

What is the cube of a^2-x?

$$\begin{array}{rl}
 & a^2-x \\
 & a^2-x \\ \hline
 & a^4-a^2x \\
 & \quad -a^2x+x^2 \\ \hline
\text{Square} = & a^4-2a^2x+x^2 \\
 & a^2-x \\ \hline
 & a^6-2a^4x+a^2x^2 \\
 & \quad -a^4x+2a^2x^2-x^3 \\ \hline
\text{Cube} = & a^6-3a^4x+3a^2x^2-x^3
\end{array}$$

It is very well known that the value of the figures in the common arithmetical scale increases in a tenfold proportion from the right to the left; a number, therefore, may be expressed by the addition of the units, tens, hundreds, &c. of which it consists.

A number of 2 figures may be expressed by $10a+b$.

............3 figures................by $100a+10b+c$.

............4 figures............by $1000a+100b+10c+d$.

EVOLUTION.

55. Evolution is the reverse of Involution, and consists in finding the square, cube, &c. roots of any given quantity.

Case I. *To extract the roots of a simple quantity, or powers.*

Rule. Extract the root of the numerical coefficient, if it have any, as in common arithmetic; then divide the index of the given power by 2, for the square root, 3 for the cube root, 4 for the biquadrate root, and depending on the index of the root required; thus the square root $9x^2=3x^{\frac{2}{2}}=3x^1=3x$; and the cube root of $8x^6=2x^{\frac{6}{3}}=2x^2$.

If the coefficient be a fraction, extract the root both of its numerator and denominator; thus, the square $\frac{4}{9}x^2=\frac{2}{3}x$, or $\frac{1}{2}x^2=\frac{1}{4}x$.

1. Find the square root of $16a^2x^2=16\times a^2\times x^2$, or
$\sqrt{16a^2x^2}=\sqrt{16}\times\sqrt{a^2}\times\sqrt{x^2}=\pm4\times a\times x=\pm4ax$, Ans.

2. Find the cube root of $8a^3x^3$, or $(8\times a^3\times x^3)$.
$\therefore\ \sqrt[3]{8a^3x^3}=\sqrt[3]{8}\times\sqrt[3]{a^3}\times\sqrt[3]{x^3}=2\times a\times x=2ax$, Ans.

3. Find the cube root of $-125a^3x^6$, or $-125\times a^3\times x^6$.
$\therefore\ \sqrt[3]{-125a^3x^6}=\sqrt[3]{-125}\times\sqrt[3]{a^3}\times\sqrt[3]{x^6}=-5\times a\times x^2=-5ax^2$ Ans.

4. Find the 5th root of $-\frac{32x^{10}y^5}{243a^{15}}$.

$$\sqrt[5]{-\frac{32x^{10}y^5}{243a^{15}}}=-\frac{2x^{\frac{10}{5}}y^{\frac{5}{5}}}{3a^{\frac{15}{5}}}=-\frac{2x^2y}{3a^3}, \text{ Ans.}$$

CASE II. *To extract the square root of a compound quantity.*

RULE. Arrange the terms according to the power of some letter, as in division. Find the root of the first term, and set it in the quotient. Subtract the square of the root thus found from the first term, and there will be no remainder. Then bring down the next two terms for a dividend, and take double (or two times) the root already found for a divisor. Divide the dividend by the divisor, and put the result both in the quotient and divisor. Then proceed as in common arithmetic.

$$\begin{array}{ll}
 & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c \\
 & a^2 \\ \hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\ \hline
2a+2b+c) & 2ac+2bc+c^2 \\
 & 2ac+2bc+c^2 \\ \hline
\end{array}$$

$$\begin{array}{l|l}
1. & 9x^4-12x^3+16x^2-8x+4(3x^2-2x+2 \\
 & 9x^4 \\ \hline
6x^2-2x & -12x^3+16x^2 \\
\text{Divisor.} & -12x^3+4x^2 \\ \hline
6x^2-4x+2 & 12x^2-8x+4 \\ \hline
\end{array}
\qquad
\begin{array}{l}
2 \\ \hline
6x^2 \text{ constant trial divisor.}
\end{array}$$

First, we extract the square root of $9x^4$, the first term. This gives $3x^2$ for the first term of the root required. This we place on the right hand of the dividend, in the manner of the quotient, as in division. Squaring this term, and subtracting it from the dividend, we get for a remainder $-12x^3+16x^2-8x+4$: we now double $3x^2$, and place it as a divisor on the left of this remainder, and dividing by it, $6x^2$, the first term of the remainder, we get or obtain the quotient $-2x$, the second term of the root sought, which we annex, with the proper sign, to the double root $6x^2$; multiply the whole of this quantity, $6x^2-2x$, by $-2x$, and subtracting the product from the first remainder, we obtain for a second remainder $12x^2-8x+4$; then by doubling $3x^2-2x$, the two terms of the root thus found, and dividing $12x^2$, the first term of the new remainder, by $6x^2$, the first term of the double root, we obtain $+2$ for a quotient, which is the third term of the root sought; and by annexing it to the double root $6x^2-4x$, and then multiplying the whole of this quotient, $6x^2-4x+2$, by 2, and subtracting the product from the second remainder, we find 0 for a new remainder, which shows that the root required is $3x^2-2x+2$.

1. $4x^6+12x^5+5x^4-2x^3+7x^2-2x+1(2x^3+3x^2-x+1$

Mult. $4x^6$ $\qquad$ 2

$4x^3+3x^2 \mid 12x^5+5x^4$ $\qquad$ $4x^3$ constant trial divis.

Mult. $\mid 12x^5+9x^4$

$4x^3+6x^2-x \mid -4x^4-2x^3+7x^2$

$\mid -4x^4-6x^3+x^2$

$4x^3+6x^2-2x+1 \mid 4x^3+6x^2-2x+1$

$\mid 4x^3+6x^2-2x+1$

Dividend.

2. $4x^3-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$

$4x^3$

$4x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}})-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}$ $\qquad$ $2x^{\frac{3}{2}}-5x^{\frac{3}{4}}a^{\frac{2}{3}}-6y^{\frac{5}{6}}b^{\frac{1}{2}}$ root.

1st divisor. $\qquad$ $-20x^{\frac{9}{4}}a^{\frac{2}{3}}+25x^{\frac{3}{2}}a^{\frac{4}{3}}$

$4x^{\frac{3}{2}}-10x^{\frac{3}{4}}a^{\frac{2}{3}}+6xy^{\frac{5}{6}}b^{\frac{1}{2}})+24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{5}{3}}$

2d divisor. $\qquad$ $24x^{\frac{3}{2}}y^{\frac{5}{6}}b^{\frac{1}{2}}-60x^{\frac{3}{4}}y^{\frac{5}{6}}a^{\frac{2}{3}}b^{\frac{1}{2}}+36by^{\frac{9}{3}}$

3. Extract the square root of $1\pm x$.

$1\pm x)1\pm\frac{1}{2}x-\frac{1}{8}x^2\pm\frac{1}{16}x^3-\frac{5}{128}x^4\pm$, &c.

1

$2\pm\frac{1}{2}x)\pm x$ $\qquad$ $(-\frac{1}{16}x^4\times 4=-\frac{4}{64}x^4$

$\pm x+\frac{1}{4}x^2$ $\qquad$ $+\frac{1}{64}x^4$

$2\pm x-\frac{1}{8}x^2)-\frac{1}{4}x^2$ $\qquad$ $*\ -\frac{5}{64}x^4$

$-\frac{1}{4}x^2\mp\frac{1}{8}x^3+\frac{1}{64}x^4$

$2+x-\frac{1}{4}x^2+\frac{1}{16}x^3)\pm\frac{1}{8}x^3-\frac{1}{64}x^4$

$\pm\frac{1}{8}x^3+\frac{1}{16}x^4\mp x^5+\frac{1}{256}x^6$

$2\pm x-\frac{1}{4}x^2\pm\frac{1}{8}x^3-\frac{5}{128}x^4)*-\frac{5}{64}x^4\pm\frac{1}{64}x^5-\frac{1}{256}x^6$

4. Extract the square root of $a^2 \pm x^2$.

$$\begin{array}{l}
a^2 \pm x^2 \,\Big|\, a \pm \frac{x^2}{2a} - \frac{x^4}{8a^3} \pm \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \pm, \text{ \&c. answer.} \\
a^2 \\
\hline
2a \pm \frac{x^2}{2a} \,\Big|\, \pm x^2 \\
\qquad \pm x^2 + \frac{x^4}{4a^2} \\
\hline
2a \pm \frac{x^2}{a} - \frac{x^4}{8a^3} \,\Big|\, - \frac{x^4}{4a^2} \\
\qquad\qquad - \frac{x^4}{4a^2} \mp \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
\hline
2a \pm \frac{x^2}{a} - \frac{x^4}{4a^3} \pm \frac{x^6}{16a^4} \,\Big|\, \pm \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
\qquad\qquad\qquad \pm \frac{x^6}{8a^4} + \frac{x^8}{16a^6} \mp \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}} \\
\hline
2a \pm \frac{x^2}{a} - \frac{x^4}{4a^8} \pm \frac{x^6}{8a^5} - \frac{5x^8}{128a^7} \,\Big|\, - \frac{5x^8}{64a^6} \pm \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}, \text{ \&c. and} \\
\hline
\end{array}$$

so on to an infinite number of terms.

$$\begin{array}{l}
1+1)1 \pm \tfrac{1}{2} - \tfrac{1}{8} \pm \tfrac{1}{16} - \tfrac{5}{128} \pm, \text{ \&c.} \\
1 \\
\hline
2 \pm \tfrac{1}{2}) \pm 1 \\
\qquad \pm 1 + \tfrac{1}{4} \\
\hline
2 \pm 1 - \tfrac{1}{8}) - \tfrac{1}{4} \\
\qquad\qquad - \tfrac{1}{4} \mp \tfrac{1}{8} + \tfrac{1}{64} \\
\hline
\qquad\qquad \pm \tfrac{1}{8} - \tfrac{1}{64}, \text{ \&c.} \\
\hline
\end{array}$$

To extract the Cube Root of a Compound Quantity.

56. Rule. Arrange the terms according to the dimensions of some letter, as in division, and extract the root of the first term, which must be a cube. Place this root in the quotient, subtract its cube from the first term, and there will be no remainder. Bring down the three next terms for a dividend, and put three times the square just found in the divisor's place, and see how often it is contained in the first term of the dividend, and the quotient is the next term of the root. Add three times the product of the two terms of the root, plus the square of the last term, to the terms already in the divisor's place, and the divisor will be complete. Multiply the divisor by the last term of the root, subtract the product from the dividend, and bring down the next three terms for a dividend, and proceed as before.

$$\begin{array}{r}
a^3+3a^2b+3ab^2+b^3(a+b \\
a^3 \qquad\qquad\qquad\quad \\
\hline
3a^2+3ab+b^3)3a^2b+3ab^2+b^3 \\
3a^2b+3ab^2+b^3 \\
\hline
\end{array}$$

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1.} \\
\multicolumn{2}{r}{x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
3x^4-6x^3+4x^2 & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-8x^3 \\
\hline
3x^4-12x^3+15x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1 \\
 & 3x^4-12x^3+15x^2-6x+1 \\
\hline
\end{array}$$

2. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^6+6x^5-40x^3+96x-64(x^2+2x-4} \\
\multicolumn{2}{r}{x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
\hline
3x^4+6x^3+4x^2 & 6x^5-40x^3 \\
 & 6x^5+12x^4+8x^3 \\
\hline
3x^4+12x^3-24x+16 & -12x^4-48x^3+96x-64 \\
 & -12x^4-48x^3+96x-64 \\
\hline
\end{array}$$

If the root consists of three terms, *a*, *b*, *c*, they may be obtained by first finding *a* and *b*, as above, and then deriving *c* from $(a+b)$ in the same manner that *b* was derived from *a*.

$$\begin{array}{rl}
\multicolumn{2}{r}{(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3(a+b+c} \\
\multicolumn{2}{r}{(a+b)^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
\hline
3(a+b)^2+3(a+b)c+c^2) & 3(a+b)^2c+3(a+b)c^2+c^3 \\
 & 3(a+b)^2c+3(a+b)c^2+c^3 \\
\hline
\end{array}$$

3. Extract the cube root of $8x^3+36x^2+54x+27$. Ans. $2x+3$.

4. Extract the cube root of $27x^6-54x^5+63x^4-44x^3+21x^2-6x+1$. Ans. $3x^2-2x+1$.

SIMPLE EQUATIONS.

1. An equation is a proposition which declares the equality of 2 quantities, expressed algebraically. This is done by connecting these quantities by the sign $(=)$: thus, $x-4=6-x$ is an equation expressing the equality of the quantities $x-4$, and $6-x$. Also, $x-5=0$ is an equation which asserts that $x-5$ is equal to nothing, and therefore that the positive part of the expression is equal to the negative part.

(2.) A simple equation is one which, when cleared of fractions and surds, contains only the first power of the unknown quantity. (See Definitions 3 and 4, page 1 ; and also Definitions 32 to 36, page 4.)

(6.) A cubic equation, or an equation of three dimensions, is one into which the cube of the unknown quantity enters, with the simple and quadratic powers.

(7.) A pure quadratic is one into which only the square of the unknown quantity enters.

(8.) An adfected quadratic is one which involves the square of the unknown quantity, and also the simple power and unknown quantities. Thus, $ax^2+b=0$ is a pure quadratic, and $ax^2+bx+c=0$ is an adfected quadratic.

(9.) The resolution of equations is the determining from some quantities given the values of others which are unknown, so that these latter may answer certain conditions proposed.

(10.) And these values are called roots of the equation.

(11.) If equal quantities be added to equal quantities, the sums will be equal.

(12.) If equal quantities be taken from equal quantities, the remainders will be equal.

(13.) If equal quantities be multiplied by the same or equal quantities, the products will be equal.

(14.) If equal quantities be divided by the same or equal quantities, the quotients will be equal.

(15.) If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

(16.) If a quantity be both multiplied and divided by another, its value will not be altered.

(17.) Any quantity may be transposed from one side of an equation to the other, by changing its sign :

Because, in this transposition, the same quantity is merely subtracted from each side of the equation; and if equals be taken from equals, the remainders are equal. Thus, if $x+9=15$, and 9 be subtracted from each side, $x=15-9$, or 6. Also, if $x+b=a$, and b be subtracted from each side, $x=a-b$. And if $x-c=d$, and c be added to each side, $x=d+c$.

Also, if $5x-7=2x+2$, and $2x$ be taken from each side, $5x-2x-7=2$, or $3x-7=2$; and if -7 be subtracted, or (which is the same thing) if $+7$ be added to each side, $3x=2+7=9$.

Also, if $x-a+b=c-3x$, then, by subtracting $-a+b-3x$ from each side, we have $x+3x=a-b+c$.

Cor. 1. Hence, if the signs of all the terms on each side of an equation be changed, the two sides still remain equal; because in this change every term is transposed.

Cor. 2. Hence, when the known and unknown quantities are connected in an equation by the signs + or —, they may be separated by transposing the known quantities to one side, and the unknown to the other.

Cor. 3. Hence, also, if any quantity be found on both sides of an equation, it may be taken away from each ; thus, if $x+y=5+y$, then $x=5$. If $a-b=c+d-b$, then $a=c+d$.

(18.) If every term on each side of an equation be multiplied by the same quantity, the results will be equal :

Because, in multiplying every term on each side by any quantity, the value of the whole side is multiplied by that quantity; and (13) if equals be multiplied by the same quantity, the products will be equal. Thus, if $x = 5 + a$, then $6x = 30 + 6a$, by multiplying every term by 6.

Cor. 1. Hence an equation, of which any part is fractional, may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction. If there be more fractions than one in the given equation, it may be so reduced by multiplying every term either by the product of the denominators, or by a common multiple of them; and if the least common multiple be used, the equation will be in its lowest terms. Thus, if $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x = 13$; if every term be multiplied by 12, which is the least common multiple of 2, 3, 4; $6x + 4x + 3x = 156$.

Cor. 2. Hence, also, if every term on both sides have a common multiplier or divisor, that common multiplier or divisor may be taken away; thus, if $ax^2 + abx = cdx$; each term being divided by the common multiplier x, $ax + ab = cd$. Also, if $\frac{5}{4}x + \frac{a+6}{4} = \frac{4x-7}{4}$, then also $5x + a + 6 = 4x - 7$.

Also, if $\frac{ax + ab}{c} = \frac{ad}{c} + \frac{4ax}{c}$, then, multiplying by $\frac{c}{a}$, $x + b = d + 4x$.

Cor. 3. Also, if each member of the equation have a common divisor, the equation may be reduced by dividing both sides by that common divisor;

Thus, if $ax^2 - a^2x = abx - a^2b$, each side is divisible by $ax - a^2$, whence $x = b$.

Cor. 4. Hence also any term of an equation may be made a square, by multiplying all the terms of the equation by the quantities necessary; as, if $ax^2 + bcx = cd^2$, the first term may be made a square by multiplying each term by a, and $a^2x^2 + abcx = acd^2$.

(19.) If each side of an equation be raised to the same power, the results are equal; thus, if $x = 6$, $x^2 = 36$;

If $x + a = y - b$, then $x^2 + 2ax + a^2 = y^2 - 2by + b^2$.

And if the same roots be extracted on each side, the results are equal; thus, if $x^2 = 49$, $x = 7$.

If $x^3 = a^3b^3$, then $x = ab$; if $x^2 + 2x + 1 = y^2 - y + \frac{1}{4}$, then $x + 1 = y - \frac{1}{2}$, and if $x^2 - 4ax + 4a^2 = y^2 + 6by + 9b^2$, then $x - 2a = y + 3b$.

For (13 and 14) when equal quantities on each side of an equation are multiplied or divided by equal quantities, the results will be equal.

Cor. Hence, if that side of the equation which contains the unknown quantity be a perfect square, cube, or other power, by extracting the square root, cube root, &c. of both sides, the equation will be reduced to one of lower dimensions: thus, if $x^2+8x+16=36$, $x+4=6$;

If $x^3+3x^2+3x+1=27$, $x+1=3$;

If $x^4+2x^3+x^2=100$, $x^2+x=10$.

(20.) Any equation may be cleared of a single radical quantity by transposing all the other terms to the contrary side, and raising each side to the power denominated by the surd. If there are more than one surd, the operation must be repeated. Thus, if $x=\sqrt{(ax+b^2)}$, by squaring each side $x^2=ax+b^2$, which is free from surds. Also, if $\sqrt{(x^2+7)}+x=7$, then, (17) by transposition, $\sqrt{(x^2+7)}=7-x$; and (19) by squaring each side, $x^2+7=49-14x+x^2$, which is free from surds. Also, if $x+\sqrt[3]{(a^2x)}=b$, then, (17) by transposition, $\sqrt[3]{(a^2x)}=b-x$; and, (19) by cubing each side, $a^2x=b^3-3b^2x+3bx^2-x^3$, which is free from surds. Also, if $\sqrt{\{x^2+\sqrt{(x^2+21)}\}}-1=x$, then, (17) by transposition, $\sqrt{\{x^2+\sqrt{(x^2+21)}\}}=x+1$, and (19) by squaring each side, $x^2+\sqrt{(x^2+21)}=x^2+2x+1$; therefore, (17, Cor. 3,) $\sqrt{(x^2+21)}=2x+1$, and, (19) by squaring each side, $x^2+21=4x^2+4x+1$, which is free from surds. And, if $\sqrt[3]{\{a^2x+\sqrt{(a^3x^3)}\}}=c$, (19) by cubing each side, $a^2x+\sqrt{(a^3x^3)}=c^3$, and (17) by transposition, $\sqrt{(a^3x^3)}=c^3-a^2x$; therefore, (19) by squaring each side, $a^3x^3=c^6-2a^2c^3x+a^4x^2$, which is free from surds.

(21.) Any proportion may be converted into an equation; for the product of the extremes is equal to the product of the means.

Let $a : l :: c : d$, by the nature of proportion $\frac{a}{b}=\frac{c}{d}$; therefore, (18, Cor. 1,) $ad=bc$.

RULE I. Any term may be transposed or transferred from one side of an equation to the other, by changing its signs.

Thus, if $x+3=7$; then will $x=7-3$, or $x=4$.

And, if $x-4+6=8$; then will $x=8+4-6=6$.

Also, if $x-a+l=c-d$; then will $x=a-b+c-d$.

From this rule it also follows, that if a quantity be found on each side of an equation, with the same sign, it may be left out of both of them; and that the signs of all the terms of any equation may be changed from + to —, or from — to +, without altering its value.

Thus, if $x+5=7+5$; then, by cancelling, $x=7$.

And if $a-x=b-c$; then, by changing the signs, $x-a=c-b$, or $x=a+c-b$.

1. Given $2x+3=x+17$ to find x. Ans. $x=$[illegible]

By transposing, gives $2x-x=17-3=14$. Whence $x=14$, Ans.

2. Given $5x-9=4x+7$ to find x. Ans. $x=$[illegible]

By transposing, gives $5x-4x=7+9=16$. Whence $x=16$, Ans.

3. Given $x+9-2=4$ to find x. Ans. $x=-3$.
By transposing, gives $x=4+2-9=-3$, Ans.

4. Given $9x-8=8x-5$ to find x. Ans. $x=3$.
By transposing, gives $9x-8x=8-5=3$. Whence $x=3$, Ans.

5. What number is that, to the double of which if 18 be added the sum will be 82?

Let $x=$ the number required. Then $2x+18=82$; $\therefore$ by transposition, $2x=64$, and $x=32$.

6. What number is that, to the double of which if 44 be added, the sum is equal to four times the required number?

Let $x=$ the number. Then $2x+44=4x$; $\therefore$ by transposition, $44=2x$, and $22=x$.

7. What number is that, the double of which exceeds its half by 6?

Let $x=$ the number. Then $2x-\frac{1}{2}x=6$; $\therefore$ $4x-x=12$, or $3x=12$, $\therefore$ $x=4$.

8. From two towns which are 187 miles distant, two travellers set out at the same time with an intention of meeting. One of them goes 8 miles and the other 9 miles a day. In how many days will they meet?

Let $x=$ the number of days required; then $8x=$ the number of miles one travelled, and $9x=$ the number the other travelled; and since they meet, they must together have travelled the whole distance, consequently $8x+9x=187$, or $17x=187$, $\therefore$ $x=11$.

Rule II. If an unknown quantity has a coefficient, its value may be found by dividing both sides of the equation by that coefficient, because if two equal quantities be divided by the same or equal quantities, the quotients will be equal.

Thus, if $ax=3ab-c$; then will $x=3b-\frac{c}{a}$.

And, if $2x+4=16$; then will $x+2=8$, or $x=8-2=6$.

1. A cask which held 146 gallons was filled with a mixture of brandy, wine, and water. In it there were 15 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Let $x=$ the number of gallons of brandy, $\therefore$ $x+15=$ number of gallons of wine, and $2x+15=$ number of gallons of water.

$\therefore$ $x+x+15+2x+15=146$, $\therefore$ by transposition, $4x=116$, and $x=29$. $\therefore$ there were 29, 44, and 73 gallons respectively of brandy, wine, and water.

2. A person employed 4 workmen; to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

Let $x=$ the sum received by the fourth,
$\therefore x+4=$ - - - - - - third,
$x+7=$ - - - - - - second,
and $x+9=$ - - - - - - first.
$\therefore x+x+4+x+7+x+9=32$,
and, by transposition, $4x=12$,
consequently $x=3$.
$\therefore$ they received 12, 10, 7, and 3 shillings respectively.

3. What number is that, the treble of which increased by 12 shall as much exceed 54 as that treble is below 144?

Let $x=$ the number. $\therefore 3x+12-54=144-3x$, $\therefore 6x=186$, and $x=31$.

4. Two persons began to play with equal sums of money: the first lost 14 dollars, the other won 24 dollars, and then the second had twice as many dollars as the first. What sum had each at first? Let $x=$ the sum;

Then $x-14$ } = the sums each had after playing;
and $x+24$ } $\therefore 2x-28=x+24$; $\therefore x=52$.

5. At a certain election 943 men voted, and the candidate chosen had a majority of 65. How many voted for each?

Let $x=$ the number of votes the unsuccessful candidate had;
$\therefore x+65=$ the number the successful one had. $\therefore x+x+65=943$; $2x=878$, and $x=439$. $\therefore$ the numbers were 439 and 504.

Rule III. Any equation may be cleared of fractions by multiplying each of its terms, successively, by the denominators of those fractions, or by multiplying both sides by the products of all the denominators, or by any quantity that is divisible by each of the denominators.

It also appears, from this rule, that if the same number, or quantity, be found in each of the terms of an equation, either as a multiplier or divisor, it may be expunged from all of them, without altering the result.

Thus, if $ax=ab+ac$; then, by cancelling, $x=b+c$.

And if $\frac{x}{a}+\frac{b}{a}=\frac{c}{a}$; then, $x+b=c$, or $x=c-b$.

1. Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$ to find x. Ans. $x=13$.

Multiplying by 12, we get $6x+6+4x+8=192-3x-9$, or $13x=169$; and, by division, $x=\frac{169}{13}=13$, Ans.

2. Given $\frac{x+a}{b}+\frac{x}{c}=\frac{2x}{a}+\frac{a+b}{d}$ to find x.

Here $\frac{x+a}{b}+\frac{x}{c}=\frac{2x}{a}+\frac{a+b}{d}$; then, multiplying by b, $x+a$

$+\frac{bx}{c}=\frac{2bx}{a}+\frac{ab+b^2}{d}$, mult. by c, $cx+ca+bx=\frac{2bcx}{a}+\frac{abc+cb^2}{d}$, and, by multiplying by a and d successively, we have $adcx+a^2cd+abdx=2bcdx+a^2bc+acb^2$, or, by transposition, $adcx+abdx=2bcdx+a^2bc+acb^2-a^2cd$, and, by division, $x=\frac{a^2bc+acb^2-a^2cd}{adc+abd-2bcd}$, A.

Rule IV. If the unknown quantity be a surd, transpose the rest of the terms, and let the surd quantity stand alone on one side of the equation; then take away the radical sign from it, and raise the other side of the equation to the power denoted by the index of the surd.

Thus, if $\sqrt{x}-2=3$; then will $\sqrt{x}=3+2=5$, or, by squaring, $x=5^2=25$.

And if $\sqrt{(3x+4)}=5$; then will $3x+4=25$, or $3x=25-4=21$, or $x=\frac{21}{3}=7$.

Also, if $\sqrt[3]{(2x+3)}+4=8$; then will $\sqrt[3]{(2x+3)}=8-4=4$, or $2x+3=4^3=64$, and $\therefore$ $2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$.

1. Given $2\sqrt{x}+3=9$ to find x. Ans. $x=9$.

Here $2\sqrt{x}+3=9$; then, $2\sqrt{x}=9-3=6$ by transposition, and $4x=36$ by squaring, or $x=\frac{36}{4}=9$, Ans.

2. Given $\sqrt{(x+1)}-2=3$ to find x. Ans. $x=24$.

Here $\sqrt{(x+1)}-2=3$; then $\sqrt{(x+1)}=3+2=5$, or $x+1=25$ by squaring, and consequently $x=25-1=24$, Ans.

3. Given $\sqrt{(4a^2+x^2)}=\sqrt{(4b^4+x^4)}$ to find x.

Here $\sqrt{(4a^2+x^2)}=\sqrt{(4b^4+x^4)}$; then, by squaring $4a^2+x^2=\sqrt{(4b^4+x^4)}$, and squaring again, $16a^4+8a^2x^2+x^4=4b^4+x^4$, by transposition and division $x^2=\frac{b^4-4a^4}{2a^2}$, and $\therefore$ $x=\sqrt{(\frac{b^4-4a^4}{2a^2})}$, An.

Rule V. If that side of the equation which contains the unknown quantity be a complete power, the equation may be reduced to a lower dimension, by extracting the root of the said power from both sides of the equation.

1. Given $9x^2-6=30$ to find x. Ans. $x=2$.

Then $9x^2=30+6=36$, or $x^2=\frac{36}{9}=4$, and $\therefore$ $x=\sqrt{4}=2$, Ans.

2. Given $x^3+9=36$ to find x. Ans. $x=3$.

Then $x^3=36-9=27$, or $x=\sqrt[3]{27}=3$, Ans.

Rule VI. Any proportion may be converted into an equation by making the products of the extremes and means equal to each other.

1. Given $10-x : \frac{2}{3}x :: 3 : 1$ to find x. Ans. $x=3\frac{1}{3}$.

Then $10-x=2x$ by mult. ext. and means, and $3x=10$, or $x=\frac{10}{3}=3\frac{1}{3}$, Ans.

2. Given $8+8x : 4x :: 8 : 2$ to find x. Ans. $x=1$.

[illegible]re $8+8x:4x::8:2$; then $16+16x=32x$ by mult. ext. and [illegible]ns, and $32x-16x=16$, or $16x=16$; $\therefore x=\frac{16}{16}=1$, Ans.

3. Given $x:6-x::2:4$ to find x. Ans. $x=2$.

Here $x:6-x::2:4$; then $4x=12-2x$ by mult. ext. and means, and $4x+2x=12$, or $6x=12$; $x=\frac{12}{6}=2$, Ans.

4. Given $4x:a::9\sqrt{x}:9$ to find x. Ans. $x=\frac{a^2}{16}$.

Then $36x=9a\sqrt{x}$ by mult. ext. and means, or $4x=a\sqrt{x}$, and by squaring $16x^2=a^2x$; therefore, by division, $x=\frac{1}{16}a^2$, Ans.

5. Given $\frac{1}{2}\sqrt{(x^2+3a^2)}-\frac{1}{2}\sqrt{(x^2-3a^2)}=x\sqrt{a}$, to find the value of x.

Multiplying the given equation by 2, we have $\sqrt{(x^2+3a^2)}-\sqrt{(x^2-3a^2)}=2x\sqrt{a}$; $\therefore$ by squaring $2x^2-2\sqrt{(x^4-9a^4)}=4ax^2$, or $-\sqrt{(x^4-9a^4)}=2ax^2-x^2$; squaring again, $x^4-9a^4=4a^2x^4-4ax^4+x^4$, or $(4a-4a^2)x^4=9a^4$, or $x=\sqrt[4]{\frac{9a^3}{4-4a}}$.

6. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, to find the value of x.

Here $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, by squaring, $2a+2\sqrt{(a^2-x^2)}=b^2$, or $\sqrt{(a^2-x^2)}=\frac{1}{2}b^2-a$; whence $a^2-x^2=\frac{1}{4}b^4-ab^2+a^2$, and $x=\sqrt{(ab^2-\frac{1}{4}b^4)}=\frac{1}{2}b\sqrt{(4a-b^2)}$.

7. Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, to find the value of x.

By cubing both sides after the form of Ex. 7, Case x, Surds, we have $2a+3\sqrt[3]{(a^2-x^2)}\{\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}\}=b^3$. But since $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, this becomes $2a+3\sqrt[3]{(a^2-x^2)}\times b=b^3$, or $3\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{b}$, or $\sqrt[3]{(a^2-x^2)}=\frac{b^3-2a}{3b}$; whence $a^2-x^2=\frac{(b^3-2a)^3}{27b^3}=(\frac{b^3-2a}{3b})^3$; therefore $x=\sqrt{\{a^2-(\frac{b^3-2a}{3b})^3\}}$.

8. Given $\sqrt{(a^2+ax)}=a-\sqrt{(a^2-ax)}$, to find the value of x.

This equation transposed is $\sqrt{(a^2+ax)}+\sqrt{(a^2-ax)}=a$, squaring $2a^2+2\sqrt{(a^4-a^2x^2)}=a^2$; whence $\sqrt{(a^4-a^2x^2)}=-\frac{1}{2}a^2$; squaring again, $a^4-a^2x^2=\frac{1}{4}a^4$, or $a^2x^2=\frac{3}{4}a^4$, whence $x=\sqrt{\frac{3}{4}a^2}=\frac{a}{2}\sqrt{3}$.

9. Given $\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$, to find the value of x.

By transposition, $\sqrt{(a^2-x^2)}=a^2\sqrt{(1-x^2)}-x\sqrt{(a^2-1)}$; squaring, $a^2-x^2=a^4-a^4x^2-2xa^2\sqrt{(1-x^2)(a^2-1)}+a^2x^2-x^2$. Again, by transposition and division, $2x\sqrt{(1-x^2)(a^2-1)}=a^2-a^2x^2+x^2-1=(1-x^2)(a^2-1)$; by squaring, $4x^2(1-x^2)(a^2-1)=(1-x^2)^2(a^2-1)^2$, and $4x^2=(1-x^2)(a^2-1)$, dividing the whole by $(1-x^2)(a^2-1)$; therefore, by actually multiplying, $4x^2=a^2-a^2x^2+x^2-1$, or $3x^2+a^2x^2=a^2-1$; whence $x^2=\frac{a^2-1}{a^2+3}$; and, consequently, $x=$

10. Given $\sqrt{(x+a)}=c-\sqrt{(x+b)}$, to find the value of x. Squaring $x+a=c^2-2c\sqrt{(x+b)}+x+b$, and transposing $2c\sqrt{(x+b)}=c^2+b-a$, or $\sqrt{(x+a)}=\frac{c^2+b-a}{2c}$; whence $x+b=(\frac{c^2+b-x}{2c})^2$, and $x=(\frac{c^2+b-a}{2c})^2-b$, the answer.

11. Given $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt{\frac{4bc}{a^2-x^2}}$, to find the value of x.

By squaring gives $\frac{b}{a+x}+\frac{c}{a-x}+\sqrt{\frac{4bc}{a^2-x^2}}=\sqrt{\frac{4bc}{a^2-x^2}}$, whence $\frac{b}{a+x}+\frac{c}{a-x}=0$, or $\frac{ab-bx+ac+cx}{a^2-x^2}=0$; therefore $ab-bx+ac+cx=0$, and $bx-cx=ab+ac$; consequently $x=\frac{ab+ac}{b-c}=a(\frac{b+c}{b-c})$.

Of the resolution of Simple Equations, containing Two Unknown Quantities.

When there are two unknown quantities, and two independent simple equations involving them, they may be reduced to one, by any of the three following rules:

RULE I. Observe which of the unknown quantities is the least involved, and find its value in each of the equations, as if the other were known, by the methods already explained; then let the two values, thus found, be put equal to each other, and there will arise a new equation, with only one unknown quantity in it, the value of which may be found as before.

1. Given $4x+y=34$, and $4y+x=16$, to find the values of x and y. Ans. $x=8$, $y=2$.

From 1st $x=\frac{34-y}{4}$, and from 2d, $x=16-4y$; therefore, by equality, $\frac{34-y}{4}=16-4y$, and, consequently, $34-y=64-16y$, or $16y-y=64-34$, or $15y=30$; whence $y=\frac{30}{15}=2$, and $x=16-8=8$.

2. Given $\left\{\begin{matrix}\frac{1}{2}x+\frac{1}{3}y=9 \\ x:y::4:3\end{matrix}\right\}$ to find x and y. Ans. $x=12$, and $y=9$.

From the 1st $x=18-\frac{2}{3}y$, and from the 2d $x=\frac{4}{3}y$, hence $\frac{4}{3}y=18-\frac{2}{3}y$, or $4y=54-2y$; whence $y=\frac{54}{6}=9$, and $x=\frac{4}{3}y=\frac{36}{3}=12$.

3. Given $x+y=80$, and $\frac{2}{3}x=\frac{3}{4}y$, to find x and y.

Here $x+y=80$, and $\frac{2}{3}x=\frac{3}{4}y$; then, from the 1st, $x=80-y$, and from the 2d, $x=\frac{9}{8}y$; hence $\frac{9}{8}y=80-y$, or $9y+8y=640$, whence $y=\frac{640}{17}=37\frac{11}{17}$, and $x=80-y=80-37\frac{11}{17}=42\frac{6}{17}$.

II. Find the values of either of the unknown quantities equation in which it is the least involved; then substitute this value in the place of its equal in the other equation, and there will arise a new equation with only one unknown quantity in it.

1. Given $\left\{\begin{array}{r} x+2y=17 \\ 3x-\ y=\ 2 \end{array}\right\}$ to find the values of x and y.

From the first equation, $x=17-2y$; which value, being substituted for x, in the second, gives $3(17-2y)-y=2$, or $51-6y-y=2$, or $7y=51-2=49$, whence $y=\frac{49}{7}=7$, and $x=17-2y=3$.

2. Given $\left\{\begin{array}{r} x+y=13 \\ x-y=\ 3 \end{array}\right\}$ to find the values of x and y.

From the first equation, $x=13-y$; which value being substituted for x, in the second, gives $13-y-y=3$, or $2y=13-3=10$, whence $y=\frac{10}{2}=5$, and $x=13-y=8$.

3. Given $\left\{\begin{array}{c} x:y::a:b \\ x^2+y^2=c \end{array}\right\}$ to find the values of x and y.

Here the analogy in the first, turned into an equation, gives $bx=ay$, or $x=\frac{ay}{b}$, and this value, substituted for x in the second, gives $(\frac{ay}{b})^2+y^2=c$, or $\frac{a^2y^2}{b^2}+y^2=c$, whence we have $a^2y^2+b^2y^2=b^2c$, or $y^2=\frac{b^2c}{a^2+b^2}$; and consequently $y=b\sqrt{\frac{c}{a^2+b^2}}$, and $x=a\sqrt{\frac{c}{a^2+b^2}}$.

4. Given $x^2y+xy^2=30$, and $x^3+y^3=35$, to find x and y.

Here $x^2y+xy^2=30$, and $x^3+y^3=35$; then, adding 3 times the 1st equation to the 2d, gives $x^3+3x^2y+3xy^2+y^3=125$, and by extracting the cube root $x+y=5$, and, from the 1st, $(x+y)xy=30$; hence, by substitution, $5xy=30$, or $xy=6$, and $x^2+2xy+y^2=25$, from which subtract 4 times the last equation, and we shall have $x^2-2xy+y^2=1$, and by evolution $x-y=1$; adding this equation to $x+y=5$, and we shall have $2x=6$, or $x=3$, whence $y=2$.

5. Given $\frac{3x-5y}{2}=\frac{2x+y}{5}-3$, and $8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$ to find x and y. Ans. $x=12$, and $y=6$.

Here $\frac{3x-5y}{2}=\frac{2x+y}{5}-3$, and $8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$; then from the 1st, $27y-11x=30$, and, from the 2d, $9x-2y=96$. Multiplying the 1st of these equations by 2, and the second by 27, we shall have $54y-22x=60$, and $243x-54y=2592$; then, by adding the two last equations together, we have $221x=2652$, or $x=\frac{2652}{221}=12$, and $9x-2y=96$; hence $2y=9x-96=108-96=12$, hence $y=\frac{12}{2}=6$.

Of the resolution of simple equations, containing three or more unknown quantities.

When there are three unknown quantities, and three independent simple equations containing them, they may be reduced to one by the following method.

Rule. III Find the values of one of the unknown quantities in each of the three given equations, as if all the rest were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in the former case; and thence the value of the third.

Or, multiply each of the equations by such numbers, or quantities, as will make one of their terms the same in them all; then having subtracted any two of these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

And in nearly the same way may four, five, &c. unknown quantities be exterminated from the same number of independent simple equations; but, in cases of this kind, there are frequently shorter and more commodious methods of operation, which can only be learnt from practice.

1. Given $\left\{\begin{array}{l}2x+4y-3z=22\\4x-2y+5z=18\\6x+7y-z=63\end{array}\right\}$ to find x, y, and z.

Multiplying the first equat. by 2,	$4x+8y-6z=44$
and subtracting the second,	$4x-2y+5z=18$
there results (A)	$10y-11z=26$
Again, mult. the first equation by 3,	$6x+12y-9z=66$
and subtracting the third,	$6x+7y-z=63$
there results	$5y-8z=3$
and mult. this result by 2,	$10y-16z=6$
which subtracted from equation (A),	$10y-11z=26$
gives	$5z=20$

$\therefore z=4$, $y=(\frac{3+8z}{5})=7$, and $x=(\frac{22-4y+3z}{2})=3$.

2. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I had 72 guineas left. What had I at first?

Let the number of guineas $=x$; then $x-\frac{1}{4}x-\frac{3}{20}x=72$; or mult. by 20, $20x-5x-3x=1440$; whence $12x=1440$, or $x=\frac{1440}{12}=120$ guineas.

SOLUTION OF QUESTIONS.

To solve a simple equation containing but one unknown quantity.

Rule. Clear the equation of fractions by rule 3, and of radicles by rule 4; second, transpose the unknown terms or quantities to one side of the equation, and the known terms to the other, by rule 1.

Collect each side into one term, and the unknown quantities, with a known coefficient, will form one side of the equation, and a known quantity the other side. Divide each side by the coefficient of the unknown quantity, and the value of the unknown will be exhibited.

1. What number is that, from the treble of which if 18 be subtracted, the remainder is 6? Ans. 8.

Let $x=$ the number; $\therefore 3x-18=6$, or $3x=24$, and $x=8$.

2. What number is that, the double of which exceeds four-fifths of its half by 40? Ans. 25.

Let $x=$ the number; $\therefore 2x-\frac{4}{5}\cdot\frac{x}{2}=40$, and $10x-2x=200$, or $x=25$.

3. In fencing the side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards, and the other 6 per day. How many days did they work? Ans. 30.

Let x, $9x$ and $6x=$ the number of days and of yards fenced by each respectively; $\therefore (9x+6x)=15x=450$, and $x=30$, Ans.

4. A mercer bought 4 pieces of silk, which together measured 50 yards; the second was twice, the third three times, and the fourth four times as long as the first. What was the respective lengths of the pieces? Ans. 5, 10, 15, 20 yards.

Let x, $2x$, $3x$, $4x$, and $10x=50$, be the number of yards in the first, 2nd, third, fourth, and the equation, respectively, and $x=5$.

5. A farmer sold 13 bushels of barley at a certain price; and afterwards 17 bushels at the same rate; and at the second time received 36 shillings more than at the first. What was the price of a bushel? Ans. 9 shillings.

Let $x=$ the price of a bushel, $13x$ and $17x$ the sum received for the first and second; $\therefore (17x-13x=4x)=36$, and $x=9$, Ans.

6. A person bought 198 gallons of beer, which exactly filled 4 casks; the first held twice as much as the second, the second twice as much as the third, and the third three times as much as the fourth. How many gallons did each hold?

Let x, $3x$, $6x$, and $12x=$ the number of gallons the fourth, third, second and first, respectively; $\therefore 22x=198$, and $x=9$, and 108, 54, 27 and 9 were the answers.

7. A silversmith has 3 pieces of metal, which together weigh 48 ounces. The second weighs 12 ounces more than the first, and the third 9 ounces more than the second. What were their respective weights? Ans. 5, 17, and 26 ounces.

Let x, $x+12$, $x+21=$ the number of ounces the first, second, and third weighed, respectively, and $3x+33=48$, or $x+11=16$, and $x=5$, and 17, 26 ounces, Ans.

8. A vinter fills a cask, containing 96 gallons, with a mixture of brandy, wine, and water. There are 20 gallons of water more than of brandy, and 17 more of wine than of water. How many are there of each?

Let x, $20+x$, $37+x$ be the number of gallons of brandy, of water, and of wine, respectively; $57+3x=96$, or $19+x=32$, and $x=13$, and the number of gallons of brandy, water, and wine, were, 13, 33, 50, respectively.

9. A gentleman buys 4 horses; for the second of which he gives $12 more than for the first; for the third $6 more than for the second; and for the fourth $2 more than for the third. The sum paid for all was $230. How much did each cost?

Let x, $x+12$, $x+18$, $x+20$ denote the price of the first, second, third, and fourth; $\therefore 4x+50=230$, or $4x=180$, and $x=45$.

10. A poor man had 6 children, the eldest of which could earn 7*d.* a week more than the second; the second 8*d.* more than the third; the third 6*d.* more than the fourth; the fourth 4*d.* more than the fifth; and the fifth 5*d.* more than the youngest. They altogether earned 10*s.* 10*d.* a week. How much could each earn a week? Ans. 38, 31, 23, 17, 13, and 8 pence per week.

Let x, $x+5$, $x+9$, $x+15$, $x+23$, and $x+30$ be the sum earned by the youngest, fifth, fourth, third, second, and eldest; $\therefore 6x+82=130$, or $x=8$.

11. An express set out to travel 240 miles in 4 days, but in consequence of the badness of the roads he found that he must go 5 miles the second day, 9 the third, and 14 the fourth day, less than the first. How many miles must he travel each day?

Let x, $x-5$, $x-9$, $x-14=$ the number of miles travelled the first, second, third, and fourth day; $\therefore 4x-28=240$, or $x-7=60$, or $x=67$.

12. There are 5 towns, in the orders of the letters A, B, C, D, E. From A to E is 80 miles. The distance between B and C is 10 miles more, between C and D is 15 miles less, and between D and E 17 miles more than the distance between A and B. What are the respective distances? Ans. from A to B 17; from B to C 27; from C to D 2; and from D to E 34 miles.

Let x, $x+10$, $x-15$, $x+17$ denote the distance from A to B, B to C, C to D, and D to E, and we have $4x+12=80$, or $x=17$.

13. A gentleman gave \$27 to two poor persons; but he gave \$5 more to one than to the other. What did he give to each?

Let x, and $x+5$ be the number of dollars given to one and the other, and $2x+5=27$, or $x=11$, and 16, are the answers.

14. What number is that, the treble of which is as much above 40, as its half is below 51? Ans. 26.

Let $2x=$ the number; $\therefore 6x-40=51-x$, and $7x=91$, or $x=13$, $2x=26$.

15. Two workmen received the same sum for their labor; but if one had received \$15 more, and the other \$9 less, then one would have had just three times as much as the other. What did they receive? Ans. 21 dollars each.

Let $x=$ the sum; $\therefore x+15=3(x-9)$, and $2x=42$, or $x=21$ dollars, Ans.

16. Two merchants entered into a speculation, by which one gained 54 dollars more than the other. The whole gain was 49 dollars less than three times the gain of the less. What were the gains? Ans. \$103, and \$157.

Let x, $x+54$ denote the less gain and the greater, and $2x+54=3x-49$.

17. The perimeter of a triangle is 75 feet, and the base is 11 feet longer than one of the sides, and 16 feet longer than the other. Required their respective lengths.

Let x, $x-11$, $x-16$ be the length of the base and the length of the sides, and $(x+x-11+x-16)=3x-27=75$; $\therefore x=34$, and the sides were 23 and 18.

18. A company settling their reckoning at a tavern, pay 8 shillings each; but observe, that if there had been 4 more, they should only have paid 7 shillings each. How many were there?

Let x and $8x$ be the number and the sum paid, and $8x=7(x+4)$; $\therefore x=28$.

19. Divide the number 46 into two such parts, that one of them being divided by 7 and the other by 3, the quotients may together be equal to 10. Ans. 18 and 28.

Let x, $46-x=$ one part and the other, and $\frac{x}{3}+\frac{46-x}{7}=10$, and $x=18$.

20. A certain sum is to be raised upon two estates, one of which pays 19 shillings less than the other; and if 5 shillings be added to treble the less payment, it will be equal to twice the greater. What are the sums paid? Ans. 33, and 52 shillings.

Let x, $x+19$ denote the payment of the less and greater in shillings, and $3x+5=2x+38$, and $x=33$, and $x+19=52$ shillings, the Ans.

21. Having bought a certain quantity of brandy at 19 shillings

a gallon, and a quantity of rum exceeding that of the brandy by 9 gallons, at 15 shillings a gallon, I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each? Ans. 34 of brandy, and 43 of rum.

Let x and $x+9$ be the number of gallons of brandy and of rum, and $19x$ and $15(x+9)$ are the prices respectively; $\therefore$ $19x=15x+135+1$, and $4x=136$, or $x=34$ gallons of brandy, and 43 of rum, Ans.

22. Two persons, A and B, have each an annual income of 400 dollars. A spends every year 40 dollars more than B, and at the end of 4 years the amount of their savings is equal to one year's income of either. What does each spend annually?

Let x, and $x+40$ denote the sum spent by B and A, and $400-x$, $360-x$, denote the sum saved by B and A; whence, $4(760-2x)=100$, or $2x=660$ and $x=330=$B, and $370=$A.

23. A draper sold two pieces of cloth, by one of which he lost 6 dollars more than by the other; and his whole loss was 5 dollars less than treble the less loss. What were the losses sustained by each piece? Ans. \$11, and \$17.

Let x, $x+6$ denote the loss sustained by one and the other, and $2x+6=3x-5$, and $x=11$, and $x+6=17$, Ans.

24. A person engaged to reap a field of corn for 4 dollars an acre, but leaving 6 acres not reaped, he received 40 dollars. Of how many acres did the field consist? Ans. 16.

Let x denote the number of acres, and $4(x-6)=40$ dollars, or $4x-24=40$, or $4x=64$, and $x=16$ acres, Ans.

25. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist? Ans. 32.

Let x, $x+7$, $x-2$ denote the number sunk, taken and burnt; then $x+x+7+x-2+15=8x$, and $5x=20$, or $x=4$, and the fleet 32.

26. A farmer hires a farm for \$$175\frac{3}{4}$ a year; part of which he is not allowed to plough; he gives \$2 per acre for the arable, and for the rest, which was 5 acres less, he gives \$$1\frac{1}{4}$ per acre. How many acres were arable, and how many not?

Let x, and $x-5$ denote the number of acres of arable and the rest; then $8x+5(x-5)=703$, or $13x=728$, and $x=56$ arable and 51 the rest.

27. A cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute? Ans. 12.

Let x, $x+10$, and $x-5$ denote the number passing through the third, first, and second, respectively; then $20(3x+5)=820$.

28. There is a number consisting of two digits, the second o which is greater than the first; and if the number be divided b the sum of its digits, the quotient is 4; but if the digits be inver ed, and that number divided by a number greater by 2 than th difference of the digits, the quotient becomes 14. Required th number.

Let $\overline{10x+y}$ be the numb.; then $\frac{10x+y}{x+y}=4$, $\frac{10y+x}{y-x+2}=14$; b the question, from the first, $10x+y=4x+4y$, or $2x=y$, and fro the 2d, $10y+x=14y-14x+28$, or $15x-4y=28$, or $(15x-8x=$ $7x=28$, or $x=4$, and $y=8$; and the number is 48, Ans.

29. A and B have certain sums of money; says A to B give me \$15 of your money, and I shall have 5 times as much a you will have left; says B to A, give me \$5 of your money, an I shall have exactly as much as you will have left. What su of money had each?

Let $x=$ A's money, $y=$ B's, then $x+15=$ what A woul have after receiving \$15 from B, $y-15=$ what B would hav left. Again, $y+5=$ what B would have after receiving \$5 fro A, $x-5=$ what A would have left.

$$\text{Hence, by the question, } \left.\begin{array}{r} x+15=5\times(y-15)=5y-75, \\ \text{and } y+5=x-5. \end{array}\right\}$$

$$\text{By transposition, } \left.\begin{array}{r} 5y-x=\ \ 90\ (A) \\ \text{and } y-x=-10\ (B) \end{array}\right\}$$

Subtract (B) from (A), $4y \quad = 100$; $\therefore y=25$, B's money.

From equation (B), $x=y+10=25+10=35$, A's money.

30. Three men began talking about their money; the fir said to the others, two-thirds of your money would make min \$37; and the second said to the others, three-fifths of your mone would make mine \$37; but, if the other two lent five-eighths o their money to the third, it would make his money amount to \$37 What sum did each possess?

Putting $a=\$37$, and z, y, x for the 1st, 2d and 3d person's dollars respectively, we shall have, by the question,

$$z+\tfrac{2}{3}y+\tfrac{2}{3}x=a, \text{ or } 3z+2y+2x=3a,$$
$$y+\tfrac{3}{5}z+\tfrac{3}{5}x=a, \text{ or } 3z+5y+3x=5a,$$
$$x+\tfrac{5}{8}z+\tfrac{5}{8}y=a, \text{ or } 5z+5y+8x=8a.$$

Taking the 1st equation from the 2d, we have $3y+x=2a$, o $x=2a-3y$; this substituted in the 1st and 3d, we have

$\left.\begin{array}{r} 4y-3z=\ a \\ 19y-5z=8a \end{array}\right\}$; from 3 times the latter of these equations subtract 5 times the former, and there will be obtained $37y=19a$. hence $y=\frac{19a}{37}=\$19$. Then $x=2a-3y=\frac{17a}{37}=17$ dolls., an

$z=\frac{4y-a}{3}=\frac{13a}{37}=\$13.$

General Solution. 1st. Let there be given the two equations, $\left\{\begin{matrix} ax+by=c \\ a'x+b'y=c' \end{matrix}\right\}$ to find the values of x and y.

Multiplying the first by a', and the second by a, they become

$$aa'x+a'by=ca'$$
$$aa'x+ab'y=c'a;$$

and, subtracting the first from the second, we have

$$(ab'-a'b)y=c'a-ca',\text{ or }y=\frac{c'a-ca'}{ab'-a'b}.$$

In like manner, if the first equation be multiplied by b', and the second by b, and the latter product subtracted from the first, we have $(ab'-a'b)x=cb'-c'b$, or $x=\frac{cb'-c'b}{ab'-a'b}$.

In this case, where there are only two unknown quantities, it is evident what factors will render the coefficients of x and y, in the two equations, equal to one another, as above; but, when there are more unknown quantities than two, it is not so obvious what factors will answer; therefore, in what follows, I shall make use of a method for determining them, which is applicable to all cases, whatever may be the number of equations.

In the preceding example, if we multiply by the indeterminate factors m and n, the equations become

$$amx+bmy=cm$$
$$a'nx+b'ny=c'n$$

and, subtracting the second equation from the first, we have

$$(am-a'n)x+(bm-b'n)y=cm-c'n.$$

Now, in order that y may disappear from this equation, $bm-b'n$ must be $=0$, then $x=\frac{cm-c'n}{am-a'n}$. But $bm-b'n$ will be equal 0, when $m=b'$ and $n=b$; and, these values being substituted for m and n in the expression for x, we have $x=\frac{cb'-c'b}{ab'-a'b}$.

Again, to make x disappear from the equation, $am-a'n$ must be $=0$, then $y=\frac{cm-c'n}{bm-b'n}$. But $am-a'n=0$, when $m=a'$ and $n=a$; therefore, substituting for m and n their values, we have $y=\frac{ca'-c'a}{ba'-b'a}$; or, changing the signs in the numerator and denominator, $y=\frac{ac'-a'c}{ab'-a'b}$.

When one of the unknown quantities has been determined by this method, the other may be found, without repeating the process, by only changing the coefficients, or writing a in place of b, and a' in place of b', or the contrary. Thus, if in the value for x we write a for b, and a' for b', it becomes $\frac{ca'-c'a}{ba'-b'a}$, the value of y. And if in the value for y we write b for a, and b' for a', we have $\frac{cb'-c'b}{ab'-a'b}$, for the value of x. 2nd. Let there be three equations, $\left\{\begin{array}{l} ax+by+cz=d \\ a'x+b'y+c'z=d' \\ a''x+b''y+c''z=d'' \end{array}\right\}$ to find the values of x, y and z.

Multiplying by the indeterminate factors m, n, and p, we have

$$\begin{array}{l} amx+bmy+cmz=dm \\ a'nx+b'ny+c'nz=d'n \\ a''px+b''py+c''pz=d''p, \end{array}$$

and, subtracting the last from the sum of the other two, there remains $(am+a'n-a''p)x+(bm+b'n-b''p)y+(cm+c'n-c''p)z=dm+d'n-d''p$. Now, in order that y and z may disappear from this equation, and x only remain,

$$\begin{array}{l} bm+b'n-b''p=0, \text{ or } bm+b'n=b''p \\ cm+c'n-c''p=0, \text{ or } cm+c'n=c''p; \end{array}$$

then $x=\frac{dm+d'n-d''p}{am+a'n-a''p}$. From the equations $bm+b'n=b''p$ and $cm+c'n=c''p$, we have, by the first case, $m=\frac{p(c'b''-b'c'')}{bc'-b'c}$, and $n=\frac{p(bc''-b''c)}{bc'-b'c}$ and since p is arbitrary, we may take it equal to the denominator $bc'-b'c$, then $m=c'b''-b'c''$, and $n=bc''-b''c$. These values being substituted for m, n and p, in the expression for x, we have

$$x=\frac{d(c'b''-b'c'')+d'(bc''-b''c)-d''(bc'-b'c)}{a(c'b''-b'c'')+a'(bc''-b''c)-a''(bc'-b'c)}.$$

In like manner, if we make $am+a'n=a''p$
and $cm+c'n=c''p$,
x and z will disappear from the equation, and we shall have $y=\frac{dm+d'n-d''p}{bm+b'n-b''p}$. And, proceeding as before, we find $m=c'a''-a'c''$, $n=ac''-ca''$, and $p=ac'-ca'$; therefore,

$$y=\frac{d(c'a''-a'c'')+d'(ac''-ca'')-d''(ac'-ca')}{a(c'a''-a'c'')+a''(ac''-ca'')-a''(ac'-ca')}.$$

Lastly, if we make $\left.\begin{array}{r} am+a'n=a''p \\ \text{and } bm+b'n=b''p \end{array}\right\}$, x and y will disappear from the equation, and we shall have $z=\frac{dm+d'n-d''p}{cm+c'n-c''p}$. And, substituting for m, n and p, their values found as above, namely,

$m=b'a''-a'b''$, $n=ab''-ba''$, and $p=ab'-ba'$, we have

$$z=\frac{d(b'a''-a'b'')+d'(ab''-ba'')-d''(ab'-ba')}{c(b'a''-a'b'')+c'(ab''-ba'')-c''(ab'-ba')}.$$

These values being multiplied out, and the terms so arranged as to be positive and negative alternately, we have

$$x=\frac{db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}.$$

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$

$$z=\frac{ab'd''-ad'b''+da'b''-ba'd''+bd'a''-db'a''}{ac'b''-ab'c''+ca'b''-ba'c''+bc'a''-cb'a''},$$

the signs in the first and third expressions being changed, in order that the denominator of each may be the same.

When one of the unknown quantities is determined, the others may be found without going through the separate operations, by only changing the coefficients; thus, if in the first value of x, we substitute the respective coefficients of y in place of those of x, viz. b in place of a, b' in place of a', and b'' in place of a'', we shall have the value of y: and if we substitute the coefficients of z instead of those of x, viz. c in place of a, c' in place of a', and c'' in place of a'', we shall have the value of z.

The reason of this will be obvious, if we suppose the unknown quantities y and z to change places successively with x in the original equations. To apply these formulæ to the example in question, where

$$\begin{aligned}2x+2y+3z&=3a\\ 3x+5y+2z&=5a\\ 8x+5y+5z&=8a\end{aligned}$$

we have $a=2$, $a'=3$, $a''=8$, $b=2$, $b'=5$, $b''=5$, $c=3$, $c'=3$, $c''=5$, $d=3a$, $d'=5a$, and $d''=8a$.

And substituting these values in the above expressions,

$$x=\frac{3\cdot5\cdot5-3\cdot3\cdot5+3\cdot5\cdot5-2\cdot5\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}{2\cdot5\cdot5-2\cdot3\cdot5+3\cdot3\cdot5-2\cdot3\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}\times a=\frac{-17}{-37}\cdot a=\frac{17a}{37}$$

$$y=\frac{2\cdot5\cdot5-2\cdot3\cdot8+3\cdot3\cdot8-3\cdot3\cdot5+3\cdot3\cdot8-3\cdot5\cdot8}{2\cdot5\cdot5-2\cdot3\cdot5+3\cdot3\cdot5-2\cdot3\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}\times a=\frac{-19}{-37}\cdot a=\frac{19a}{37}$$

$$z=\frac{2\cdot5\cdot8-2\cdot5\cdot5+3\cdot3\cdot5-2\cdot3\cdot8+2\cdot5\cdot8-3\cdot5\cdot8}{2\cdot5\cdot5-2\cdot3\cdot5+3\cdot3\cdot5-2\cdot3\cdot5+2\cdot3\cdot8-3\cdot5\cdot8}\times a=\frac{-13}{-37}\cdot a=\frac{13a}{37}$$

as before.

This method may be extended with the same facility to as many unknown quantities as we please; thus, if there be four equations

$$ax+by+cz+dw=e$$

$$a'x+b'y+c'z+d'w=e'$$
$$a''x+b''y+c''z+d''w=e''$$
$$a'''x+b'''y+c'''z+d'''w=e'''$$; then, multiplying the first by m, the second by n, the third by p, the fourth by q, and subtracting the last product from the sum of the other three, we find $(am+a'n+a''p-a'''q)x+(bm+b'n+b''p-b'''q)y+(cm+c'n+c''p-c'''q)z+(dm+d'n+d''p-d'''q)w=em+e'n+e''p-e'''q$. Therefore, to find x, we have $m+b'n+b''p-b'''q=0$, $cm+c'n+c''p-c'''q=0$ and: $dm+d'n+d''p-d'''q=0$, and $x=\dfrac{em+e'n+e''p-e'''q}{am+a'n+a''p-a'''q}$.

The values of m, n and p must now be found from the formulæ in case 2nd, and substituted in the last expression, we shall then have the value of x; and from thence, by changing the coefficients, we shall obtain the values of y, z and w. The values of x, y, z, and w, are as below. The numerator of x is

$$eb'c''d'''-eb'd''c'''+ed'b''c'''-ed'c''b'''+ec'd''b'''-ec'b''d'''$$
$$+bd'c''e'''-bd'e''c'''+bc'e''d'''-bc'd''e'''+be'd''c'''-be'c''d'''$$
$$+ce'b''d'''-ce'd''b'''+cd'e''b'''-cd'b''e'''+cb'd''e'''-cb'e''d'''$$
$$+db'e''c'''-db'c''e'''+dc'b''e'''-dc'e''b'''+de'c''b'''-de'b''c'''$$

The numerator of y is

$$ae'c''d'''-ae'd''c'''+ad'e''c'''-ad'c''e'''+ac'e''d'''-ac'd''e'''$$
$$+ed'c''a'''-ed'a''c'''+ec'a''d'''-ec'd''a'''+ea'd''c'''-ea'c''d'''$$
$$+ca'e''d'''-ca'd''e'''+cd'a''e'''-cd'e''a'''+ce'd''a'''-ce'a''d'''$$
$$+de'a''c'''-de'c''a'''+dc'e''a'''-dc'a''e'''+da'c''e'''-da'e''c'''$$

The numerator of z is

$$ab'e''d'''-ab'd''e'''+ad'b''e'''-ad'e''b'''+ae'd''b'''-ae'b''d'''$$
$$+db'e''a'''-bd'a''e'''+be'a''d'''-be'd''a'''+ba'd''e'''-ba'e''d'''$$
$$+ea'b''d'''-ea'd''b'''+ed'a''b'''-ed'b''a'''+eb'd''a'''-eb'a''d'''$$
$$+db'a''e'''-db'e''a'''+de'b''a'''-de'a''b'''+da'e''b'''-da'b''e'''$$

The numerator of w is

$$ab'c''e'''-ab'e''c'''+ae'b''c'''-ae'c\,'b'''+ac'e''b'''-ac'b''e'''$$
$$+be'c''a'''-be'a''c'''+bc'a''e'''-bc'e''a'''+ba'e''c'''-ba'c''e'''$$
$$+ca'b''e'''-ca'e''b'''+ce'a''b'''-ce'b''a'''+cb'e''a'''-cb'a''e'''$$
$$+eb'a''c'''-eb'c''a'''+ec'b''a'''-ec'a''b'''+ea'c''b'''-ea'b''c'''$$

And the denominator of each is

$$ab'c''d'''+ab'd''c'''+ad'b''c'''+ad'c''b'''+ac'd''b'''-ac'b''d'''$$
$$+bd\,c''a'''-bd'a''c'''+bc'a''d'''-bc'd''a'''+ba'd''c'''-ba'c''d'''$$
$$+ca\,b''d'''-ca'd''b'''+cd'a''b'''-cd'b''a'''+cb''d''a'''-cb'a''d'''$$
$$+db'a''c'''-db'c''a'''+dc'b''a'''-dc'a''b'''+da'c''b'''-da'b''c'''$$

If there were 5 unknown quantities, the numerators and denominators of each would consist of 120 terms.

31. Given $\{x+\frac{1}{2}(y+z)=17$, $y+\frac{1}{3}(x+z)=17$; and $z+\frac{1}{4}(x+y)=17\}$, or $\{x+\frac{1}{4}(y+z)=1000$; $y+\frac{1}{5}(x+z)=1000$; and $z+\frac{1}{6}(x+z)=1000\}$; to find x, y and z, in both the above cases.

If $a=2$; $b=3$; $c=4$; Ans. $x=5$, $y=11$, $z=13$.

A general theorem for all questions of a similar nature:

$$\frac{(abc+b+c-2bc-a)n}{abc+2-a-b-c}=\frac{67n}{107}=\text{A's.}\ \frac{(abc+a+c-2ac-b)n}{abc+2-a-b-c}=\frac{77n}{107}$$

$$=\text{B's.}\quad\text{and}\ \frac{(abc+a+b-2ab-c)n}{abc+2-a-b-c}=\frac{83n}{107}=\text{C's part.}$$

In this question n is $=$ to one thousand.

EXPRESSION OF QUESTIONS.

As it is sometimes difficult for learners to know how to express the conditions of a question algebraically, the following remarks may be found useful:

If the question be concerning one unknown number, or quantity; it may be represented by x or y.

If the unknown quantity is to be multiplied by 5, that condition is expressed thus: $5x$, or $5y$.

If 4 be added to that product, and the sum is equal to 14, then, $5x+4=14$; but if 4 is to be subtracted, then $5x-4=14$.

If the unknown quantity is to be divided by 3, that condition may be expressed thus: $\frac{x}{3}$ or $\frac{y}{3}$; and if 7 be subtracted from that quotient, and the remainder is equal to 10, those conditions are expressed thus: $\frac{1}{3}x-7=10$.

If the third, fourth, and fifth of a number be added to itself, and the sum is equal to 35, that condition is expressed thus:

$$x+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=35.$$

If the question is concerning two numbers or quantities, they may be called x and y.

If it be required that the sum of the two numbers sought be 60, that condition is expressed thus: $x+y=60$.

If their difference must be 24, then $x-y=24$.

If their product is 156, then $xy=156$.

If their quotient must be 6, then $\frac{x}{y}=6$, or $x\div y=6$.

If their ratio is as 3 to 2, then $x:y::3:2$, and hence $2x=3y$.

If the sum of their squares is 100, then $x^2+y^2=100$.

If the difference of their squares is 27, then $x^2-y^2=27$.

If the product of their squares is 36, then $x^2y^2=36$.

If the quotient of their squares is 16,

Then $\frac{x^2}{y^2}=16$, or $x^2\div y^2=16$.

If their sum added to their difference be 10, and x be the greater number and y the less, then $2x=10$;

For $x+y+x-y=2x$.

If the difference of their sum and difference be 6, and x be the greater and y the less, then $2y=6$;

For $x+y-(x-y)=2y$.

If the product of their sum, multiplied by their difference, must be 64, then $(x+y)(x-y)=64$; whence $x^2-y^2=64$.

If the square of their sum be 36,

Then $(x+y)^2$, or $x^2+2xy+y^2=36$.

If the square of their difference be 16,

Then $(x-y)^2$, or $x^2-2xy+y^2=16$.

If the sum of the squares of their sum and of their difference be 100, then $2x^2+2y^2=100$;

For $x^2+2xy+y^2$ added to $x^2-2xy+y^2=2x^2+2y^2$.

If the difference of the squares of the sum and difference equal 64, then $4xy=64$. As may be seen by subtracting, $x^2-2xy+y^2$ from $x^2+2xy+y^2$.

If their sum, divided by their difference, be 3, then $\frac{x+y}{x-y}=3$.

If two thirds of one, and four sevenths of the other, make 60,

$$\text{Then } \frac{2x}{3}+\frac{4y}{7}=60.$$

If two thirds of one, subtracted from four sevenths of the other, leave 14, then $\frac{4x}{7}-\frac{2y}{3}=14$.

If their sum must be four times their difference,

Then $x+y=4(x-y)$, or $x+y=4x-4y$.

If the sum of their squares is five times the sum of the numbers, then $x^2+y^2=5(x+y)=5\times(x+y)$.

Or $x^2+y^2=5x+5y$.

If their product is six times their sum,

Then $xy=6(x+y)$, or $xy=6x+6y$.

If their product is nine times their quotient, then $xy=\frac{9x}{y}$.

If one number must be as much above 20 as the other wants of 20, then $x-20=20-y$.

If one number must be three times as much above 20 as the other wants of 20, then $x-20=3(20-y)=3\times(20-y)$, Or $x-20=60-3y$.

If their difference and sum are to each other as the numbers two and three, then $x-y : x+y :: 2 : 3$; or,

Multiplying extremes and means, $3x-3y=2x+2y$.

If their sum and product are to each other as the numbers three and five, then $x+y : xy :: 3 : 5$; or,

Multiplying extrems and means, $5x+5y=3xy$.

If one number ought to be as many times contained in 20 as the other contains the number 4, then $\frac{20}{x}=\frac{y}{4}$; or $20 : x :: y : 4$; Whence $xy=80$.

If the number 20 must be a mean proportional between the two numbers sought, then $x : 20 :: 20 : y$; whence $xy=400$.

If the greater being divided by the less, and again the less by the greater, the first quotient must be to the second as 5 to 3;

Then $\frac{x}{y} : \frac{y}{x} :: 5 : 3$. Whence $\frac{3x}{y}=\frac{5y}{x}$.

If one number increased by 2, and multiplied by the other diminished by three, produce 40, then $(x+2)(y-3)=40$, or $xy-3x+2y-6=40$.

If three numbers, x, y, and z, must be in continued proportion, then $x : y :: y : z$. Whence, $xz=y^2$.

It is sometimes easier to employ fewer letters than there are unknown quantities. Thus, the solution becomes more easy and elegant. There are some examples of this kind of notation.

Conditions.	*Notation.*
The sum of the two numbers sought is 60	Let $x=$ one number, then will $60-x=$ the other.
Their difference is 24.	x and $x+24$ (or $x-24$).
Their product is 146.	x and $\frac{146}{x}$.
Their quotient is 6.	x and $\frac{x}{6}$, or x and $6x$.

Their ratio is as 3 to 2. x and $\frac{2x}{3}$, or $3x$ and $2x$; because $\frac{3x}{2x}=\frac{3}{2}$, or $3x : 2x :: 3 : 2$.

The greater is 4 times the less .... x and $4x$.

These and the preceding are some of the relations which are easily expressed; many others occur which are less obvious, but as they cannot be described by particular rules, their expression is best explained by examples, and must be acquired by experience.

On the Solution of Simple Equations which involve more than one unknown Quantity.

If the equation involve several unknown quantities, and definite values of these are required, there must necessarily be as many independent equations as there are unknown quantities. In which case, the values will be found by exterminating all the unknown quantities except one; and this may be done by either of the three following methods:

1. By equalizing the coefficients of the same unknown quantity in the several equations.
2. By substitution.
3. By equating different values of the same unknown quantity.

1. Of exterminating an unknown quantity by the first method in equations where two unknown quantities are concerned.

If the coefficient of either unknown quantity in one equation be contained a certain number of times exactly in the coefficient of the same unknown quantity in the other, multiply the former equation by that number, then add it to, or subtract it from, the other equation, according as the signs are different or the same, and an equation arises, in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{r}4x+y=34\\ \text{and } 4y+x=16\end{array}\right\}$ Here the coefficient of x in the second equation is contained 4 times exactly in the first; multiplying therefore the second equation by 4, and subtracting the first from it,

$$\begin{array}{r}4x+16y=64,\\ \text{and } \underline{4x+y=34;}\\ \therefore 15y=30, \text{ and } y=2.\end{array}$$

Having thus obtained a value of one of the unknown quantities, the other may be determined by substituting in either equation the value of the quantity found, and thus reducing the equation to one which contains only the other unknown quantity. Thus, from the second of the preceding equations, $x=16-4y=16-8=8$.

The values of x and y might be found in a similar manner, by multiplying the first equation by 4, and subtracting the second from it. But if neither of the coefficients be a measure of the coefficients of the same unknown quantity in the other equation, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of the same unknown quantity in the first. If the signs of the unknown quantity be alike in both, subtract one equation from the other; if unlike, add them together, and an equation arises in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{r}2x+3y=23\\ \text{and } 5x-2y=10\end{array}\right\}$ In this case neither of the coefficients is a measure of the coefficient of the same unknown quantity in the other equation; and therefore, multiplying the first equation by 2, and the second by 3,

$$\begin{array}{r}4x+\ 6y=46,\\ \text{and } 15x-\ 6y=30;\\ \hline\end{array}$$

∴ by addition, $19x=76$, and $x=4$; whence, as before,

$3y=23-2x=23-8=15$, and $y=5$.

The values of x and y might also be obtained, by multiplying the first equation by 5, and the second by 2, and then subtracting the second from the first.

2. By substitution. Find the value of one of the unknown quantities, in terms of the other and known quantities, in the more simple of the two equations; and substitute this value instead of the quantity itself in the other equation; thus an equation is obtained in which there is only one unknown quantity.

Thus in the first of the preceding examples; from the second equation, $x=16-4y$; substituting therefore this value of x in the first equation, $4\cdot(16-4y)+y=34$, or $64-16y+y=34$; ∴ by transposition, $(64-34=)30=15y$, and therefore $2=y$; whence, as before, $x=8$. Here a value of x might have been obtained from the second equation, and substituted for it in the first; whence an equation would have arisen, involving only y; the value of which being found, that of x also might be determined, as before, by substitution.

Or a value of y might be determined from either equation, and substituted in the other; from which would arise an equation involving only x, the value of which might be found; and therefore the value of y also might be obtained by substitution.

Again, in the second example; from the first equation is obtained $2x=23-3y$; and therefore $x=\frac{23-3y}{2}$;

substituting therefore this value in the second equation,

$5\cdot\frac{23-3y}{2}-2y=10$, or $115-15y-4y=20$; ∴ by transposition, $115-20=15y+4y$, or $95=19y$; ∴ $5=y$, and $x=\frac{23-3y}{2}=\frac{23-15}{2}=\frac{8}{2}=4$. Here also a value of x might be obtained from the second equation, and substituted in the first, which would give an equation involving only y; or a value of y might be obtained from either equation, which substituted in the other would give an equation involving only x; the value of which might therefore be found, and consequently that of y might also be determined.

3. By equating different values of the same unknown quantity.

From each equation find the value of the same unknown quantity in terms of the other and known quantities; then, by equating the values so found, an equation arises containing only one unknown quantity. Thus, in the first of the preceding examples; from the first equation, $y=34-4x$, and from the second equation, $4y=16-x$; and therefore $y=\frac{16-x}{4}$; $\therefore$ $\frac{16-x}{4}=34-4x$; consequently, $16-x=136-16x$; $\therefore$ by transposition, $16x-x=136-16$; or $15x=120$;

$\therefore$ $x=8$, and $y=34-4x=34-32=2$, as before.

In this case, also, two values of x are deducible from the two equations, which would give an equation involving y only; and the value of y being determined, that of x might also be found.

Again, in the second of the preceding examples; from the first equation, $x=\frac{23-3y}{2}$, and from the second, $x=\frac{10+2y}{5}$; therefore, $\frac{10+2y}{5}=\frac{23-3y}{2}$, and $20+4y=115-15y$; by transposition, $4x+15y=115-20$, or $19y=95$; $\therefore$ $y=5$, and $x=4$, as before. Here, again, two values of y might have been found, which would have given an equation involving only x; and from the solution of this new equation, a value of x, and therefore of y, might be found.

1. Given $\left.\begin{aligned} 5x+4y&=58 \\ \text{and } 3x+7y&=67 \end{aligned}\right\}$ to find the values of x and y.

Multiplying the second equation by 5, and the first by 3,

$$\begin{array}{r} 15x+35y=335, \\ \text{and } 15x+12y=174; \\ \hline \end{array}$$

$\therefore$ by subtraction, $23y=161$, and $y=7$;

whence, $5x=58-4y=58-28=30$, and therefore $x=6$.

If the second equation had been multiplied by 4, and subtracted from the first when multiplied by 7, an equation would have arisen, involving only x, the value of which might be determined, and thence, by substitution, the value of y.

Second Method.

From the second equation, $3x=67-7y$; $\therefore$ $x=\frac{67-7y}{3}$. Substituting this value of x in the first equation, $5\cdot\frac{67-7y}{3}+4y=58$, and $335-35y+12y=174$; $\therefore$ by transposition, $335-174=$

$35y-12y$, or $161=23y$; $\therefore 7=y$; whence, as before, the value of x may be found. In the same manner, a value of x might be found from the first equation, which substituted in the second, would give an equation involving only y. Or a value of y might be obtained from either equation, and substituted for it in the other; whence an equation would arise involving only x, the value of which might be found, and therefore that of y also determined.

Third Method.

From the first equation, $5x=58-4y$; $\therefore x=\frac{58-4y}{5}$.

From the second, $x=\frac{67-7y}{3}$. $\therefore \frac{58-4y}{5}=\frac{67-7y}{3}$, and $174-12y=335-35y$;
by transposition, $35y-12y=335-174$, or $23y=161$; $\therefore y=7$; whence, as before, $x=6$. In this case, two values of y might be deduced from the two equations; and from equating these, there would arise an equation involving x only; whose value being found, that of y also might be determined by substitution.

2. Given $\left.\begin{array}{r} ax+by=m \\ cx+dy=n \end{array}\right\}$ to find the values of x and y.

Multiplying the first equation by c, and the second by a,

$$\begin{array}{l} acx+bcy=mc, \\ \underline{acx+ady=na;} \end{array}$$

$\therefore$ by subtraction, $(ad-bc)\cdot y=na-mc$, and $y=\frac{na-mc}{ad-bc}$;

whence $x=\frac{m}{a}-\frac{by}{a}=\frac{m}{a}-\frac{nab-mbc}{a^2d-abc};=\frac{mad-mbc}{a^2d-abc}-\frac{nab-mbc}{a^2d-abc},=\frac{mad-nab}{a^2d-abc}=\frac{md-nb}{ad-bc}$. Or the value of x might be determined from the second equation, $x=\frac{n}{c}-\frac{dy}{c}$.

If the first equation had been multiplied by d, and subtracted from the second multiplied by b, an equation would have arisen involving only x, the value of which might be determined; and this being substituted in either of the equations, the value of y might also be found.

Second Method.

From the first equation, $ax=m-by$; and $\therefore x=\frac{m-by}{a}$.

Substituting this value of x in the second equation,

$c\cdot\frac{m-by}{a}+dy=n$; $\therefore mc-bcy+ady=an$, and $(aa-bc)\cdot y=an-mc$; $\therefore y=\frac{an-mc}{ad-bc}$; whence, the value of x may be determined, as before.

In the same manner, a value of x might be found from the second equation, which substituted in the first would give an equation involving only y, the value of which being found, that of x might also be determined. Or, a value of x might be obtained from either equation, which substituted in the other would give an equation involving only x, the value of which, and consequently that of y, might be found.

Third Method.

From the first equation, $x=\frac{m-by}{a}$, and from the second, $x=\frac{n-dy}{c}$; $\therefore \frac{m-by}{a}=\frac{n-dy}{c}$; and $mc-bcy=na-ady$; $\therefore$ by transposition, $ady-bcy=na-mc$;

$\therefore y=\frac{na-mc}{ad-bc}$; whence, as before, $x=\frac{md-nb}{ad-bc}$.

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise another equation involving only x, the value of which being determined, that of y also might be found by substitution.

1. A draper bought two pieces of cloth for £12, 13*s*; one being 8*s*., and the other 9*s*. per yard. He sold them each at an advanced price of 2*s*. per yard, and gained by the whole £3. What were the lengths of the pieces?

Ans. 17 yards of the first, and 13 of the second.

Let x and y be the lengths, and $8x$ and $8y=$ what they cost, and by the question $\left\{\begin{array}{l}8x+8y=253\\2x+2y=\ \ 60\end{array}\right\}\begin{array}{l}x=17\\y=13.\end{array}$

2. A bill of £26 5*s*. was paid with half guineas and crowns, and twice the number of half guineas exceeded three times the number of crowns by 17. How many were there of each?

Let $2x$ and y denote the number of half guineas and of crowns; then $\left\{\begin{array}{l}21x+5y=525\\4x-3y=\ \ 17\end{array}\right\}$ or $\left.\begin{array}{l}63x+15y=1575\\20x-15y=\ \ 85\end{array}\right\}\begin{array}{l}x=20\\y=21.\end{array}$

3. A person expends half a crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodated his neighbour with half his apples and one-third of his pears for 13 pence. How many did he buy of each?

Let $2x$ and $3y$ denote the number of apples and pears; then $\frac{1}{2}x=$ the price of apples, and $\frac{3}{5}y=$ the price of the pears, and by the question $\frac{1}{4}x+\frac{1}{5}y=13$; and $\frac{1}{2}x+\frac{3}{5}y=30$, and $2x=72, 3y=60$.

4. A and B playing at backgammon, A bet \$3 to \$2 on every game, and after a certain number of games found that he had lost \$17. Now had A won 3 more from B, the number he would then have won, would have been to the number B would have won as 5 to 4. How many games did they play?

Let x and y denote the number A and B won; then $3y-2x=17$, and $x+3 : y-3 :: 5 : 4$, and $y=7$, and $x=2$.

5. A vinter has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After they become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold? Let x and $y=$ the number of gallons each holds; then $x-15 : y-11 :: 8 : 3$, and $\frac{1}{2}x+10 : \frac{1}{2}y+10 :: 9 : 5$, by the question and from the first, $3x = 8y - 43$ and from the 2d equation, $5x = 9y + 80$; and we have $40y - 27y = 240 + 215$, or $y = \frac{455}{13} = 35$, and $x=79$.

6. A person having laid out a rectangular bowling-green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yards shorter, the ratio would have been 4 to 3. What are the lengths of the sides? Let x and $y=$ the sides; $\therefore$

$$\begin{array}{l}\text{hence } x+4 : y+4 :: 5 : 4, \\ \text{and } \quad x-4 : y-4 :: 4 : 3,\end{array}\Big\} \begin{array}{l} x+4 : x-y :: 5 : 1, x=36 \\ x-y : x-4 :: 1 : 4, y=28.\end{array}$$

$$\text{ex æquali, } x+4 : x-4 :: 5 : 4, \text{ or } x : 4 :: 9 : 1.$$

7. At an election for two member of parliament, three men offer themselves as candidates, and all the electors give single votes. The numbers of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 : to 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each?

Let $9x$, $8x$, and y be the numbers; then $8x-y : x+7 :: 7 : 12$, and $\frac{1}{2}(9x+y+1)=8x+7$; and from the first, $96x-12y=7x+49$; and from the 2d, $9x+y+1=16x+14$, and $x=\frac{205}{5}=41$; $\therefore y=7x+13=300$, and $8x=328$, $9x=369$.

8. To determine three numbers such that if 6 be added to the first and second, the sums will be in the proportion of 2 : 3; if 6 be added to the first and third, the sums will be in the proportion of 7 : 11; but if 36 be subtracted from the second and third, the remainders will be as 6 : 7.

Let $2x-6$, $3x-6$, and y be the numbers; then $2x-1 : y+5 :: 7 : 11$, and $3x-42 : y-36 :: 6 : 7$. Hence, $22x-11=7y+35$, and $7x+2y=26$, $x=\frac{90}{5}=18$, and $2y=7x-26=100$, or y

9. Two shepherds, A and B, are intrusted with the charge of 2 flocks of sheep. A's consisting chiefly of ewes, many of which produced lambs, is at the end of the year increased by 80; but B finds his stock diminished by 20; when their numbers are in the proportion of 8 to 3. Now had A lost 20 of his sheep, and had B an increase of 70, the numbers would have been in the proportion of 7 to 10. What were the numbers?

Let x and y denote the numbers; $\therefore x+80 : y-20 :: 8 : 3$, and $x-20 : y+90 :: 7 : 10$. Hence $y=\frac{6490}{59}=110$, and $x=\frac{480}{3}=160$.

10. Two persons, A and B, can perform a piece of work in 16 days. They work together for 4 days, when A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately?

Let x and y be the number of days; then by the question, $\frac{16}{x}+\frac{16}{y}=1$, and $\frac{4}{x}+\frac{4}{y}+\frac{36}{y}=1$, or $(\frac{4}{x}+\frac{40}{y}=1)-(\frac{4}{x}+\frac{4}{y}=\frac{1}{4})$, $=\{36\div y\}=\frac{3}{4}$ and $y=48$, and $x=24$.

11. There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five-twelfths of the cistern is filled in four hours; and if one of the equal cocks be stopped, seven-ninths of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

Let x and y be the numbers; $\frac{8}{x}+\frac{4}{y}=\frac{5}{12}$ and $\frac{32}{3x}+\frac{32}{3y}=\frac{7}{9}$; $\therefore (\frac{8}{x}+\frac{8}{y}=\frac{7}{12})-(\frac{8}{x}+\frac{4}{y}=\frac{5}{12})=\frac{4}{y}=\frac{1}{6}$, and $y=24$, and $\frac{8}{x}=\frac{1}{4}$, and $x=32$.

12. Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered B; in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes. How many miles does each go per hour?

Let x and y be the number the first and second goes; then $\frac{17}{x}+\frac{56}{y}=\frac{41}{3}=13\frac{2}{3}$ hours, and $\frac{147}{x}-\frac{147}{y}=28$, by the question; then $\frac{8}{7}(\frac{147}{x}-\frac{147}{y}=28)-3(\frac{17}{x}+\frac{56}{y}=\frac{41}{3})=(\frac{168}{x}-\frac{168}{y}=32)$

$-(\frac{51}{x}+\frac{168}{y}=41)=\frac{219}{x}=73$, and $\frac{3}{x}=1$ and $x=3$, and $y=\frac{21}{3}=7$.

13. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7 being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were their weights at first?

Let $4x$ and $5x=$ their weight, and $6y$ and $7y=$ the parts taken out; $\therefore 4x-6y : 5x-7y :: 2 : 3$. and $9x-13y=10$, $y=2$, $x=4$.

14. A gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12 shillings, and only half the number of men relieved, the rest being divided amongst the women, each woman would have received two shillings more than each man did. But if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

Let x and y be the number of shillings one man and one woman received; $\therefore 14x+15y=$ the whole sum, and $\frac{7x+15y-12}{15}=x+2$, and $\frac{14x+7y}{14}=2y$. Hence $x=21$, and $y=14$.

15. When wheat was 5 shillings a bushel, and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Let x and y denote the number of bushels of wheat and rye he must buy; $5x+3y=$ his money; and then by the question, $\frac{5x+3y-21}{5}+7=x+y-2$, $5x+3y-21+35=5x+5y-10$ or $24=2y$ and $30+3(x+y-6)=5x+3y=6$ hence, $y=12$, and $x=9$.

16. A draper bought two pieces of cloth of different kinds for $744; there were 6 yards of the coarser more than there were of the finer; and had the coarser cost 2 dollars a yard more than it did, 6 yards of the coarser would have cost just as much as 5 yards of the finer. He afterwards bought 4 yards of the finer, and 12 of the coarser at the same prices per yard, and found their value less than that of the former pieces in the ratio of 20 : 31. How many yards did he buy the first time, and what did he give per yard for each?

Let $x=$ the number of yards of the finer, and $y=$ the number of the coarser; $\therefore y=$ the price of a yard of the coarser, and the price of a yard of the finer $=\frac{6(y+2)}{5}$; $\therefore \frac{6x(y+2)}{5}+y(x+6)=744$, and $\frac{24(y+2)}{5}+12y : 744 :: 20 : 31$, or $\frac{24(y+2)}{5}+12y : 24 :: 20 : 1$, or $\frac{2y+4}{5}+y=40$, or $y=\frac{196}{7}=28$, and $x=\frac{576}{64}=9$.

17. A mercer bought two pieces of silk of different lengths for £50; the price of two yards of the shorter was 6*s*. 8*d*. more than the price of 3 yards of the longer; and each piece cost the same sum. He cut off two yards from each, and sold the rest for £53 12*s*. Now if he had sold the whole at that rate, he would have gained £5 by each piece. How many yards did each piece contain?

Let $x=$ the number contained in the shorter, and y that of the larger; then $\frac{25}{x}$ and $\frac{25}{y}=$ the price of a yard of each, and $\frac{50}{x}-\frac{75}{y}=\frac{1}{3}$; also, $\frac{30}{x}(x-2)+\frac{30}{y}(y-2)=53\frac{3}{5}$; $\therefore 60-\frac{60}{x}-\frac{60}{y}=53\frac{3}{5}$; whence $\{(\frac{10}{x}+\frac{10}{y}=\frac{16}{15})-(\frac{10}{x}-\frac{15}{y}=\frac{1}{15})\}=\frac{25}{y}=1$, and $y=25$, and $x=15$.

18. A sets out express from C towards D, and three hours afterwards B sets out from D towards C, travelling 2 miles an hour more than A. When they meet, it appears that the distances they have travelled are in the proportion of 13 to 15; but had A travelled five hours less, and B gone 2 miles an hour more, thay would have been in the proportion of 2 : 5. How many miles did each go per hour, and how many hours did they travel before they met?

Let $x=$ the number of miles A went per hour; $\therefore x+2=$ the number B went; let $y=$ the number of hours B travelled; $\therefore y+3=$ the number A travelled; and $x(y+3) : y(x+2) :: 13 : 15$; $\therefore 15xy+45x=13xy+26y$, or $2xy=26y-45x$. Again, $x(y-2) : y(x+4) :: 2 : 5$; $\therefore 5xy-10x=2xy+8y$, or $3xy=10x+8y$; whence, $20x+16y=78y-135x$, or $155x=62y$, and $5x=2y$; $\therefore 3xy=4y+8x=12y$, and $x=4$, $y=10$.

19. A and B engaged to reap a field of corn in 12 days. The times in which they could severally reap an acre are as 2 : 3. After some time, finding themselves unable to finish it in the stipulated time, they called in C to help them; whose rate of working was such, that if he had wrought with them from the beginning, it would have been finished in 9 days. Also the times in which he could severally have reaped the field with A alone, and with B alone, are in the proportion of 7 to 8. When was C called in?

Let $3x =$ A's daily work; $\therefore 2x =$ B's; let $z =$ C's, and $y =$ number of days C worked; $\therefore (3x+2x)12+yz =$ the whole quantity of work done; and $9(3x+2x+z) =$ the same; $\therefore 12(3x+2x)+yz=9(3x+2x+z)$, and $15x+yz=9z$. Now $3x+z : 2x+z :: 8 \cdot 7$, and $5x=z$; $\therefore 3z+yz=9z$, or $y+3=9$; $\therefore y=6$, and C was called in after 6 days.

20. The letters l, m, n representing any given numbers, as for instance, 16, 17 and 18, it is required to find a, b, c.

$$\text{Suppose } \begin{matrix}(1.)\\(2.)\\(3.)\end{matrix} \left\{ \begin{matrix} a^2+bc=l=16 \\ b^2+ac=m=17 \\ c^2+ab=n=18 \end{matrix} \right\} \text{ What are the numbers } a, b, c\,?$$

Before I proceed to the solution, I shall give Dr. Wallis's account of the problem in his own words, from which I have been induced to think that it may have been intended by the first proposer, Dr. Pell, as a trial of the skill of the author of the arithmetic of infinities, of whose algebra it occupies four whole chapters, viz. 60th, 61st, 62d, and 63d, extending over three and thirty folio pages.

"For a further explanation of Dr. Pell's method, I here subjoin another question, not undetermined as the former; but (in itself determined) proposed to myself, long since, by Col. Silas Titus, (then of his majesty's bed-chamber) a very ingenious person, and well skilled in affairs, civil and military, and very well accomplished in mathematics and other learning.

The process of which (because I understood from the colonel, it was a question proposed by Dr. Pell) I drew up in general terms, (after Dr. Pell's method, with which the colonel was well acquainted,) in this form; as I find it yet amongst my loose papers."

The 62d chapter concludes with the following paragraph:

"And thus I have pursued the problem, through all its cases and varieties, to an absolute and universal solution and determination of them. Which I have done more at large and distinctly, to shew a method how the like may be done in other problems in like manner proposed."

I think it must now appear highly probable that this was a sort of challenge from Dr. Pell to Dr. Wallis, and that the latter considered the problem worthy his best attention, and the solution as fit to be a pattern to succeeding analysts: that he retained this opinion in 1693 is a strong presumtion that his cotemporaries had given him no reason for changing it, yet this solution has been pronounced "remarkably operose and inelegant," and so it certainly is, but for patient labor, exactness and completeness almost beyond example or imitation.

By transposing a^2 in the first, and b^2 in the second equation, we obtain (4.) $bc=l-a^2$,

and (5.) $ac=m-b^2$, then by the multiplication of equals, there results (6.) $abc^2=m(l-a^2)-(l-a^2)b^2$,
and by transposing ab in the third equation, (7.) $n-ab=c^2$, whence by multiplying the two left and the two right-hand members of (6.) and (7.), and omitting the common factor c^2, we shall obtain the following quadratic in b, viz. $nab-a^2b^2=m(l-a^2)-(l-a^2)b^2$, which arranged according to the powers of b becomes (8.) $(l-2a^2)b^2+nab=m(l-a^2)$ Another equation in b and a without c will enable us to solve the problem, by finding an equation in a, and the given quantities; this we proceed to find as follows: We have seen that $bc=l-a^2$, and that $m-b^2=ac$, now multiply their left, and also their right-hand members together, and omit the common factor c, there results, for a second equation in a and b, the following cubic, viz.

(9.) $mb-b^3=(l-a^2)a$.

But by (8.) $m(l-a^2)=(l-2a^2)b^2+nab$, therefore, by multiplying their left, and also their right-hand members, and omitting the common factor $(l-a^2)b$ there will result $m^2-mb^2=a(l-2a^2)b+na^2$, aud by transposition, we obtain

(10.) $m^2-na^2=mb^2+a(l-2a^2)b$.

But by (8.) $(l-2a^2)b^2+nab=m(l-a^2)$ therefore multiplying as before, and omitting the common factor b,
$(m^2-na^2)(l-2a^2)b+na(m^2-na^2)=m^2(l-a^2)b+m(l-a^2)(l-2a^2)a$;
by transposition, $\{(m^2-na^2)(l-2a^2)-m^2(l-a^2)\}b=$
$\{m(l-a^2)(l-2a^2)-n(m^2-na^2)\}a$.
Whence, by division, we obtain the following value of b, viz.

$$(11.)\ b=\frac{a\{m(l-a^2)(l-2a^2)-n(m^2-na^2)\}}{(l-2a^2)(m^2-na^2)-m^2(l-a^2)}$$
$$=\frac{2ma^4+(n^2-3lm)a^2+m(ll-mn)}{a(2na^2-ln-m^2)}.$$

Again, by multiplying (10.) by $l-2a^2$, and (8.) by m, we obtain
$m(l-2a^2)b^2+a(l-2a^2)^2b=(l-2a^2)(m^2-na^2)$
and $m(l-2a^2)b^2+mnab=m^2(l-a^2)$, whence, by subtraction,
$\{(l-2a^2)^2-mn\}ab=(l-2a^2)(m^2-na^2)-m^2(l-a^2)$.
Whence, by division, b is found, viz.

$$(12.)\ b=\frac{(l-2a^2)(m^2-na^2)-m^2(l-a^2)}{a\{(l-2a^2)^2-mn\}}=\frac{a(2na^2-ln-m^2)}{4a^4-4la^2+l^2-mn}$$

Now by equating the values of b in (11.) and (12.) we obtain the following final equation in a, viz.

(13.) $8a^8-20la^6+2(9l^2-mn)a^4+(5lmn-7l^3-m^3-n^3)a^2+(ll-mn)^2=0$, if this equation be multiplied by 2, and then ee substituted for $2aa$, there will result
(14.) $e^8+5le^6+(9ll-mn)e^4+(5l-mn-7l^3-m^3-n^3)e^2+2(ll-mn)^2=0$,

which coincides with that found by Dr. Wallis, Algebra, pp. 230, edit. 1685, also pp. 250, lat. edit. 1693.

Now then e^2 being found from (14.) $a=\sqrt{(\frac{1}{2}e^2)}$; then b is found by (11.) or (12.), and lastly $c=\frac{l-a^2}{e}$.

When 16, 17, 18 are substituted for l, m, n in the final equation, it becomes $e^8-80e^6+1998e^4-14937e^2+5000=0$, and hence, Wallis finds the approximate value of e^2 to be 12.7564,4179,4460,744; or 0·350,987,046; or 34·832,280,28; or 32·060,290,88; and the corresponding values of a, b, c are

$a = 2\cdot525,513,986,744,158$ $\quad a = 0\cdot418,919,470$
$b = 2\cdot969,152,768,619,848$ $\quad b = 3\cdot912,226,866$
$c = 3\cdot240,580,681,617,074$ $\quad c = 4\cdot044,884,670$
$a = +\ 4\cdot173,264,926$ $\quad a = +\ 4\cdot003,766,407$
$b = +\ 4\cdot287,022,553$ $\quad b = -\ 0\cdot007,099,744$
$c = -\ 0\cdot330,331,815$ $\quad c = +\ 4\cdot245,989,3$

And he observes that if the signs of every one of any set be changed, they will solve the problem, because the signs of the planes aa, bb, cc, ab, ac, cb are not changed.

As it was not generally known, when the Doctor solved this problem, that b and c might be expressed by rational functions of a, he determines b from a quadratic equation in a, and l, m, n, as also from a cubic, and is at pains to show that only one root of the quadratic will serve, and that the cubic furnishes none serviceable but this same; c he finds from the equation $c=\frac{l-a^2}{b}$. The quadratic the Doctor found is less simple than mine; a ascending in it to the 4th power: the quadratic and cubic resulting for the extermination of b after c has been expelled are the same in Kirkby, as in this of mine; but his solution is complicated with the unnecessary factors a^2 and $l-a^2$ as that of Wallis is by the square of this last; and I think they both seem to have imagined these factors somehow connected with the general problem, though it seems evident, from the solution here given, that they are not in the least. I have only to add that Dr. Wallis's solution is, in my opinion, as curious a morsel as can easily be found, as from its copiousness and minuteness, it enables us to ascertain very exactly the analytical skill of one who made so great a figure in the world of science, during the latter half of the seventeenth century, being along with Lord Brounker, the unfoiled champion of England in the mathematical contests with M. M. Frenicle, Pascal, and Fermat, and in the doctrine of series and quadrature of curves, the precursor of Newton.

21. In one of the corners of a rectangular garden there is a fish-pond, whose area is one-ninth part of the whole garden; the periphery of the garden exceeding that of the fish-pond by 200 yards. Also, if the greater side be increased by 3 yards, and the other by 5 yards, the garden will be enlarged by 645 square yds. The fish-pond is a rectangle about the same diameter with the garden. Required the periphery of the garden, and the length of each side.

Let $x =$ the length of the lesser side, and y the length of the greater; $\therefore \frac{1}{3}x$ and $\frac{1}{3}y =$ the lengths of the greater and lesser sides of the fish-pond, (Eucl. B. 6. Prop. 24.) Also $2.(x+y) =$ the periphery of the garden; and $\frac{1}{3}\{2.(x+y)\} =$ the periphery of the fish.pond; $\therefore 2.(x+y)-\frac{2}{3}.(x+y)=200$, or $\frac{4}{3}.(x+y)=100$; $\therefore x+y=150$. Also $(y+3)\cdot(x+5)=xy+645$; $\therefore$ by transposition, $3x+5y=630$, but from the former equation, $3x+3y=450$; $\therefore$ by subtraction, $2y=180$, and $y=90$; $\therefore x=150-y=60$; and $\therefore$ the periphery $=$ 300 yards, and the sides are 60 and 90 yards.

22. A and B each bought $300 Bank stocks, A into the three per cents. and B into the Four's. These stocks were at such a price that B received one dollar interest more than A. When afterwards each of the stocks rose 10 per cent., they sold out their money, and A found himself $10 richer than B. Required the prices of the stocks.

Let $x =$ the price of the Three per cents., and $y =$ the price of the Four's; $\therefore x : 300 :: 3 :$ A's interest $= \frac{900}{x}$; and $y : 300 :: 4 :$ B's interest $= \frac{1200}{y}$; $\therefore \frac{900}{x}+1=\frac{1200}{y}$

Again, $x : x+10 :: 300 :$ what A received when selling $= \frac{300.(x+10)}{x}$; and in the same way B received $\frac{300.(y+10)}{y}$; $\therefore \frac{300\cdot(x+10)}{x}=\frac{300\cdot(y+10)}{y}+10$, and $30+\frac{300}{x}=30+\frac{300}{y}+1$; $\therefore \frac{300}{x}=\frac{300}{y}+1$, and $\frac{900}{y}+4=\left(\frac{900}{x}+1=\right)\frac{1200}{y}$ from the first equation; $\therefore$ by transposition, $4=\frac{300}{y}$, and $y=75$; $\therefore \frac{300}{x}=5$, and consequently, $x=60$; $\therefore$ the prices of the stock were 60 and 75 per cent.

23. $500 was to be lent out at simple interest in two separate sums, the smaller at 2 per cent. more than the other. The interest of the greater sum was afterwards increased, and that of the smaller sum diminished by 1 per cent. By this, the interest of the whole was augmented by one-fourth of the former value. But

if the interest of the greater sum had been so increased, without any diminution of the less, the interest of the whole would have been increased one-third. What were the sums and the rate per cent. of each ?

Let $x =$ the less sum ; $\therefore 500 - x =$ the greater ; let $y + 1 =$ the interest of the less ; $\therefore y - 1 =$ the interest of the greater, and $\frac{x.(y+1)}{100} + \frac{(500-x).(y-1)}{100} = \frac{x}{50} + 5y - 5 =$ the former interest, and $\frac{xy}{100} + \frac{(500-x)\cdot y}{100} = 5y =$ the second interest ; $\therefore 5y = \frac{5}{4}\cdot\left(\frac{x}{50} + 5y - 5\right)$, and $\therefore 4y = \frac{x}{50} + 5y - 5$; by transposition, $y = 5 - \frac{x}{50}$. Again, the 3d interest $= \frac{x\cdot(y+1)}{100} + \frac{(500-x)\cdot y}{100} = \frac{x}{100} + 5y$; $\therefore \left(\frac{x}{100} + 5y\right)\cdot 3 = 4\cdot\left(\frac{x}{50} + 5y - 5\right)$, or $\frac{3x}{100} + 15y = \frac{8x}{100} + 20y - 20$; by transposition, $\left(20 - \frac{5x}{100} =\right) 20 - \frac{x}{20} = 5y = 25 - \frac{x}{10}$, from the former equation ; $\therefore \frac{x}{20} = 5$, and $x = 100$; also $y + 1 = 6 - \frac{x}{50} = 4$; $\therefore$ the sums were 100 and 400 dollars, and the rates of interest \$4 and \$2, respectively.

24. A clock has three indices, A, B, and C. A goes round the circumference once in 12 hours, B goes round the circumference once in 1 hour, and C goes round the circumference once in one minute. How often, and at what times are they together in every twelve hours ?

Suppose x the time when A and B are together, and y the time when B and C are together ; then $x = \frac{12}{11}, \frac{2 \times 12}{11}, \frac{3 \times 12}{11}$, &c. to $\frac{12n}{11}$, by the question. But $1 : 1 : y : y =$ space B moves through, and $\frac{1}{60} : 1 : y : 60y =$ space C moves through; and because C has moved 1 circumference more than B ; $\therefore 60y - y - 1$, or $y = (\frac{1}{59}$h. $=)$ 1m.1″1‴, &c. Or $y = \frac{1}{59}, \frac{2}{59}, \frac{3}{59}, \frac{1}{59}m$, &c. Now the value of m and n in $\frac{m}{59} = \frac{12n}{11}$. That $x = y$. ; $11m = 708n$; hence $n = 11, 22$, &c., and $m = 708, 1416$, &c. But $\frac{12 \times 11}{11} = 12$; and $\frac{708}{59} = 12$. Therefore the 3 hands are together only at 12 o'clock.

On Quadratic Equations.

65. Quadratic Equations are divided into pure and adfected. Pure quadratic equations are those which contain only the square of the unknown quantity; such as $x^2=36$; $x^2+5=54$; $ax^2-b=c$; &c. Adfected quadratic equations are those which involve both the square and simple power of the unknown quantity, such as $x^2+4x=45$; $3x^2-2x=21$; $ax^2+2bx=c+d$; &c. &c.

On the Solution of Pure Quadratic Equations.

66. The Rule for the solution of pure quadratic equations is this: "Transpose the terms of the equation in such a manner, that those which contain x^2 may be on one side of the equation, and the known quantities on the other; divide (if necessary) by the coefficient of x^2; then extract the square root of each side of the equation, and it will give the value of x."

Let $x+5=54$. By transposition, $x^2=54-5=49$.

Extract the square root of both sides of the equation, then

$$x=\sqrt{49}=7.$$

Let $3x^2-4=71$. By transposition, $3x^2=71+4=75$. Divide by 3, $x^2=\frac{75}{3}=25$. Extract the square root, $x=\sqrt{25}=5$.

On the Solution of Adfected Quadratic Equations.

67. The most general form under which an adfected quadratic equation can be exhibited is $ax^2+bx=c$; where a, b, c, may be any quantities whatever, positive or negative, integral or fractional. Divide each side of this equation by a, then $x^2+\frac{b}{a}x=\frac{c}{a}$. Let $\frac{b}{a}=p$, $\frac{c}{a}=q$; then this equation is reduced to the form $x^2+px=q$, where p and q may be any quantities whatever, positive or negative, integral or fractional.

68. From the two-fold form under which adfected quadratic equations may be expressed, there arise two Rules for their solution.

69. Let $\pm ax^2 \pm bx \pm c = \pm d$ be an adfected quadratic equation, then, by transposing and dividing by $\pm a$, it becomes $x^2 \pm \frac{b}{a}x = \frac{\pm d \mp c}{\pm a}$; or, by putting p for $\frac{b}{a}$, and q for $\frac{d \mp c}{\pm a}$, it is $x^2 \pm px = \pm q$.

Add now the square of $\frac{1}{2}p$ to each side of this equation, and there results $x^2 \pm px + \frac{1}{4}p^2 = \pm q + \frac{1}{4}p^2$, where it is readily perceived that the first side is a complete square, viz. $(x \pm \frac{1}{2}p)^2$; consequently, if the square root of each side be extracted, we obtain $x \pm \frac{1}{2}p = \pm \sqrt{(\pm q + \frac{1}{4}p^2)}$, the double sign $\pm$ being placed before the radical, because the square root of a quantity may be either + or —; hence it appears that

$$x = +\sqrt{(\pm q + \tfrac{1}{4}p^2)} \mp \tfrac{1}{2}p, \text{ or } -\sqrt{(\pm q + \tfrac{1}{4}p^2)} \mp \tfrac{1}{2}p.$$

70. The above general values of x evidently include every possible case, from which separate formulæ for each distinct case are readily obtained, and are as follow:

In equations of the form,

$$\text{Case 1.}\quad x^2 + px = q,\ x = \begin{cases} +\sqrt{(q+\frac{1}{4}p^2)} - \frac{1}{2}p, \\ \text{or } -\sqrt{(q+\frac{1}{4}p^2)} - \frac{1}{2}p, \end{cases}$$

$$\text{Case 2.}\quad x^2 - px = q,\ x = \begin{cases} +\sqrt{(q+\frac{1}{4}p^2)} + \frac{1}{2}p, \\ \text{or } -\sqrt{(q+\frac{1}{4}p^2)} + \frac{1}{2}p, \end{cases}$$

$$\text{Case 3.}\quad x^2 + px = -q,\ x = \begin{cases} +\sqrt{(-q+\frac{1}{4}p^2)} - \frac{1}{2}p, \\ \text{or } -\sqrt{(-q+\frac{1}{4}p^2)} - \frac{1}{2}p, \end{cases}$$

$$\text{Case 4.}\quad x^2 - px = -q,\ x = \begin{cases} +\sqrt{(-q+\frac{1}{4}p^2)} + \frac{1}{2}p, \\ \text{or } -\sqrt{(-q+\frac{1}{4}p^2)} + \frac{1}{2}p. \end{cases}$$

In the third and fourth forms, when $(\frac{p}{2})^2$, or $\frac{p^2}{4}$ is less than $-q$, the root will be impossible, (the square root of a negative quantity being impossible, viz. $\sqrt{(-q + \frac{1}{4}p)}$) because in the solution of problems, if the expression for the root happens to be the square root of a negative quantity, it will show that the conditions, in such equations, are inconsistent. Every quadratic equaton has 2 equal or 2 unequal roots, either real or imaginary. Sometimes one root is *affirmative* and the other *negative*, and sometimes they are both *affirmative*, or both *negative*.

Rule I. Hence it appears, that if to each side of the equation be added the square of half the coefficient of the second term, there will arise, on the left-hand side of the equation, a quantity which is the square of $x + \frac{1}{2}p$; and by extracting the square root of each side of the resulting equation, we obtain a simple equation, from which the value of x may be determined.

Rule. II. *Hindoo Method of Solving Quadratics.*

71. Let the equation $ax^2 \pm bx = c$ be multiplied by $4a$, then $4a^2x^2 \pm 4abx = 4ac$, and if b^2 be added to each side, the equation becomes $4a^2x^2 \pm 4abx + b^2 = 4ac + b^2$; now the first side is evidently a square, $= (2ax \pm b)^2$, whence

$$2ax \pm b = \pm\sqrt{(4ac + b^2)},\ \therefore x = \frac{\pm\sqrt{(4ac+b^2)} \mp b}{2a}.$$

From which we infer, that if each side of the equation be multiplied by four times the coefficient of x^2, and to each side there be added the square of the coefficient of x, the quantity on the left-hand side of the equation will be the square of $2ax+b$. Extract the square root of each side of the equation, and there arises a simple equation, from which the value of x may be determined.

If $a=1$, the equation is reduced to the form $x^2+px=q$; in this case, therefore, the Rule may be applied, by multiplying each side of the equation by 4, and adding the square of the coefficient of x.

Rule Third, from Emerson's Algebra, (1764.)

When you have large numbers to deal with, it is better to proceed thus. Clear the equation, and if $a^2+ba=d$, then $a=\frac{d}{b+a}$, the form. To find the first quotient figure, take $\frac{d}{b}$, when b is far greater than a; or take $\sqrt{d}$, when a is far greater than b; or take $\frac{d}{2b}$ when a and b are nearly equal; thus it will easily be found by a few trials. Or, in general, take the first figure such, that when it is multiplied by the sum of itself and b, it will produce the first figure or figures of d, or the next less: this is all the difficulty. Then multiply and subtract as usual; the remainder is the resolvend.

Then, to continue the division, you must find a new divisor, for each quotient figure, thus. Add the last quotient figure to the last divisor, (duly observing their places,) for a new divisor; see how oft this is contained in the resolvend, set the answer in the quotient, and also add it to the divisor; then multiply the whole divisor by that quotient figure, and subtract the product for a new resolvend. But when any of the signs are negative, the proper quantities are to be subtracted, instead of being added. This work is always to be repeated for each quotient figure.

When any quotient figure is so great that the product exceeds the resolvend, place a less figure in the quotient.

When you have got more than half your intended number of figures in the quotient, you may continue the division, without adding the new quotient figures to the divisor.

Observe, each quotient figure is to be added twice to the divisor, once before multiplication, and once after; just as in extracting the square root, and for the same reason. For this method extracts the square root when $b=0$.

When one root is had, the other is found by adding this to the coefficient b; for the sum, changing its sign, is the other root.

This rule is the foundation of the method for extracting the roots of adfected equations.

Scholium. If $x^4+bx^2=d$. Put $a=x^2$, then $a^2+ba=d$; and find a as above. Then $x=\sqrt{a}$, by extracting the root. And the same for higher equations.

Let $a^2+32a=4644$, to find a. then $a=\frac{4644}{32+a}$. Suppose $\frac{4600}{32}=100$ too great for a. $\sqrt{4644}=60$, which is also too great for a. Take $a=\frac{4600}{64}=7$, too great. Take $a=50$.

$$\begin{array}{rrl}
32 & & \\
+50 & & \\
\hline
82\,) & 4644 & (50 \\
+\ 54 & 410\cdot & \\
\hline
136) & 544 & (\ 4 \\
 & 544 & 54 = a.
\end{array}$$

2. Let $a^2+35a=28349994$

$$a=\frac{28349994}{35+a}$$

Here $a=\sqrt{28}$, &c. $=5000$

$$\begin{array}{rl}
+35 & \\
5000 & \\
\hline
\left.\begin{array}{r}5035\\5300\end{array}\right\} & \begin{array}{l}28349994(5307=a\\25175\ldots\end{array} \\
\hline
\left.\begin{array}{r}10335\\307\end{array}\right\} & \begin{array}{r}31749\\31005\end{array} \\
\hline
10642) & \begin{array}{r}74494\\74494\\\hline\end{array}
\end{array}$$

3. Suppose $a^2-5307a=-184520$, to find a.

then $a=\frac{-184520}{-5307+a}=\frac{184520}{5307-a}$.

Here $a=\frac{184}{6}=30$ nearly.

$$\begin{array}{rl}
5307) & 184520(35=a. \\
-30 & 15831\cdot \\
\hline
5277) & 26210 \\
-35 & 26210 \\
\hline
5242) & \cdot\cdot
\end{array}$$

4. Let $x^2+463x=26698$, or $x=\frac{26698}{463+a}$.

$$\begin{array}{lrl}
 & \mathrm{c}\qquad r\qquad P & N \\
r+C & =463+50=513\ ) & 26698\ (\ 51.855342=x \\
 & r+s=+\ 51 & 2565 \\
\hline
 & r+P+s=564 & 1048=N' \\
 & s+t=+\ 1.8 & 564 \\
\hline
 & r+P+2s+t=565.8 & 484.0000=N'' \\
 & t+u=+\ .85 & 452.64 \\
\hline
r+ & P+2s+2t+u=566.65 & 31.3600=N''' \\
 & 5 & 28.3325 \\
\hline
 & 5|6|6.|70 & 3.0275 \\
 & & 2.8335 \\
\hline
 & & 1940 \\
 & & 1700 \\
\hline
 & & 240 \\
 & & 226 \\
\hline
 & & 14-11=3 \text{ remainder.}
\end{array}$$

On the Solution of Quadratic Equations by the Tables of Sines and Tangents.

When the numerical parts of a quadratic equation are either large numbers or complicated fractions, its solution may often be more readily obtained, by means of one of the following trigonometrical formulæ, than by the common method, which, in cases of this kind, becomes very laborious.

1. Given $x^2 + ax = b$,

$$\text{Put } \frac{2r}{a}\sqrt{b} = \text{tan. } z\text{; then } y = \left\{\begin{array}{l} +\frac{1}{r}\sqrt{b}\times\text{tan. } \frac{1}{2}z, \\ -\frac{1}{r}\sqrt{b}\times\text{cot. } \frac{1}{2}z. \end{array}\right.$$

2. Given $x^2 - ax = b$,

$$\text{Put } \frac{2r}{a}\sqrt{b} = \text{tan. } z\text{; then } x = \left\{\begin{array}{l} -\frac{1}{r}\sqrt{b}\times\text{tan. } \frac{1}{2}z, \\ +\frac{1}{r}\sqrt{b}\times\text{cot. } \frac{1}{2}z. \end{array}\right.$$

3. Given $x^2 - ax = b$,

$$\text{Put } \frac{2r}{a}\sqrt{b} = \text{sin. } z\text{; then } x = \left\{\begin{array}{l} +\frac{1}{r}\sqrt{b}\times\text{tan. } \frac{1}{2}z. \\ +\frac{1}{r}\sqrt{b}\times\text{cot. } \frac{1}{2}z. \end{array}\right.$$

In the last of which cases it is to be observed, that if $\frac{2r}{a}\sqrt{b}$ be greater than the radius r, or $4b$ greater than a^2, the two roots, or values of x, will be impossible.

1. Given $x^2+\frac{7}{44}x=\frac{1695}{12716}$, to find the roots of the equation, or the two values of x.

Here tan. $z = \frac{88r}{7}\sqrt{\frac{1695}{12716}}$

Log. 1695 . . 3.2291697

Log. 12716 . . 4.1043505

$2)\bar{1}.1248192$

Square root . $\bar{1}.5624096$

Log. 88 . . . 1.9444827

Colong. 7 . . 9.1549020

Log. tan. z . . 10.6617943

Or $z=77° 42' 32''$, and $\frac{1}{2}z=38°$ $51' 16''$, whence

Log. tan. $\frac{1}{2}z$ . . 9.9061115

Former root . . $\bar{1}.5624096$

Log. $\frac{1}{r}$ . . . $\overline{10}.0000000$

Log. x. . . . $\bar{1}.4685311$

Or . . $x=.2941176=\frac{5}{17}$ the positive root.

And if cot. $\frac{1}{2}z$ be taken, instead of tan. $\frac{1}{2}z$, the other value of x will be found $= -.4532085$; or $\frac{1695}{12716} \div -\frac{5}{17} = -\frac{339}{748}$, the negative root.

1. Given $x^2+1728\frac{1}{3}x=123578$, to find the value of x. Here the value of $x = 68.7652857$.

2. Given $x^2-\frac{127}{364}x=\frac{2547}{59825}$, to find the value of x.*

3. Given $x^2+\frac{7}{44}x=\frac{1695}{12716}$, to find the value of x.

* Ans. $x = .44464894$. Ans. $x = .29411765$.

4. Given $364x^2-127x=15.49699958$, to find x, by Emerson's method.

—127	+15.4969(—.095747
—32.76	+14.3784
—159.76 D	1.11859958
—32.76	.971700
—192.52	.14689958
—1.820	.13749036
—194.340 D	940922
—1.820	786737
—196.160	154185
—2548	137691
—196.4148 D	16496
2548	15736
—196.669|6	758
—146	580
—196.684|2 D	178
—146	177
—195.69|9	1
3	
—19|6.|7|0|2	

839 Root.

5. Given $102x^2-101x=-16.27843657$, to find the value of x.

	—101	—16.27843657
	+ 71.4	—20.72
Divisor	—29.6	+4.45
	+71.4	3.9968
	41.8	.444763
	8.16	411838
Divisor	49.96	3292543
	8.16	2979950
	58.12	312593
	714	298280
Divisor	58.834	14313
	714	11932
	59.548	2381
	51	1790
Divisor	59.599	591
	5	537
Divisor	59.656	54
Divisor	59.|6|6|1	54

(·787552399 Ans.

Rule 4. When an equation assumes the form of $x^3+cx=ab$, a solution may be effected by quadratics as often as $b^2+c=a$; for multiplying both sides by x, there is produced $x^4+cx^2=abx$, $=a\times bx$; and by adding b^2x^2 to these equals, $x^4+(b^2+c)x^2=b^2x^2+abx$; therefore, by page art. 70, I have $x^4+(b^2+c)x^2+\frac{1}{4}(b^2+c)^2=b^2x^2+abx+\frac{1}{4}(b^2+c)^2$, that is, $x^4+ax^2+\frac{1}{4}a^2=(bx)^2+a(bx)+\frac{1}{4}a^2$; and by extracting the root, $x^2+\frac{1}{2}a=bx+\frac{1}{2}a$, or $x^2=bx$, and $x=b$.

Given $x^3+3x=140$, to find x. Here 140 is a composite number, and if it be resolved into the factors 28×5, then $a=28$, $b=5$, and $c=3$, b^2+c being equal to a, therefore $x=5$.

Rule 5. When an equation assumes the form of $x^{4m}-2x^{3m}+x^{m}=a$, a solution may be effected by quadratics, in this manner: add x^{2m} to both sides of the equation, and transpose x^m; then $x^{4m}-2x^{3m}+x^{2m}=x^{2m}-x^m+a$; that is, $(x^{2m}-x^m)^2-(x^{2m}-x^m)=a$; by art. 70, case 1, I have $(x^{2m}-x^m)-\frac{1}{2}=\sqrt{(a+\frac{1}{4})}$, or $x^{2m}-x^m=\sqrt{(a+\frac{1}{4})}+\frac{1}{2}$, a quadratic equation.

7 10

Questions producing Quadratic Equations.

The methods of expressing the conditions of questions of this kind, and the consequent reduction of them, till they are brought to a quadratic equation, involving only one unknown quantity and its square, are the same as those already given for simple equations.

1. A and B gained $18 by trade. A's money was in trade 12 months, and he received for his principal and gain $26. Also B's money, which was $30, was in trade sixteen months. What money did A put into trade?

Let $x =$ the number of dollars he put in; $\therefore 26 - x =$ the number he gained, and $12x + 16 \times 30 : 12x :: 18 : 26 - x$, or, $18x = 1040 - 14x - x^2$, or $x^2 + 32x = 1040$; and, by case 1, $x = 20$, or -52; consequently A put in $20, Ans.

2. A sold a quantity of brandy for $39, and gained as much per cent. as the brandy cost him. What was the price of the brandy?

Let $x =$ the price of the brandy; then $100 : x :: x : =$ the gain, $\frac{x^2}{100}$, and $\therefore \frac{x^2}{100} = 39 - x$, or $x^2 = 3900 - 100x$, or $x^2 + 100x = 3900$; and, by case 1, $x = 30$, or -130; consequently, the price was $30, Ans.

3. A sold a piece of cloth for $24, and gained as much per cent. as it cost him. What was the price of the cloth?

Let $x =$ cost price of the cloth in dollars; then $24 - x =$ the whole gain. Hence $100 + x : x :: 24 : 24 - x$; $(100 + x) \cdot (24 - x) = 24x$; $\therefore 2400 - 100x - x^2 = 0$; $\therefore x^2 + 100x = 2400$, by case 1, we have $x =$ $20, the price of the cloth.

4. A horse dealer bought a horse for a certain number of dollars, and sold it again for $119, by which means his profit was as much per cent. as the horse cost him. Required the cost price of the horse? Ans. $70.

$100 + x : x :: x :: 119 : 119 - x$, or $x^2 + 100x = 10900$.

5. A person bought a horse which he afterwards sold for $24, and by so doing lost as much per cent. as the horse cost him. What sum did he cost? Ans. $60.

Let x denote the number of dollars the horse cost; then, as the owner lost x per cent., the loss upon x dollars will be found thus: $100 : x :: x : \frac{x^2}{100}$; since $\therefore$ he lost $\frac{x^2}{100}$, and the horse cost him x dollars, it is obvious he must have sold it for $x - \frac{x^2}{100}$; therefore $x - \frac{x^2}{100} = 24$, or $x^2 - 100x = -2400$; by case 4 we have $x^2 -$

$100x+2500=2500-2400$, or $x-50=\sqrt{100}=\pm 10$, and $\therefore$ $x=50\pm 10=60$, or 40. Hence it appears that the price of the horse might be either 60 or 40 dollars, for both these values of x equally satisfy the conditions of the question.

6. A sets out from C towards D, and travels 8 miles a day. After he had gone 27 miles, B set out from D towards C, and goes every day $\frac{1}{20}$th of the whole journey; and, after he had travelled as many days as he goes miles in one day, he met A. Required the distance of the places C and D.

Let x be the distance in miles; then B travelled $\frac{x}{20}$ miles a day, and $\frac{x^2}{400}$ miles in all. Also, A travelled $27+\frac{8}{20}x$, or $27+\frac{2x}{5}$ miles in all. By the question, $\frac{x^2}{400}+27+\frac{2x}{5}=x$, or $x^2+10800+160x=400x$; by transposition, $x^2-240x=-10800$; by case 4, $x^2-240x+(120)^2=14400-10800=3600$; root is $x-120=\sqrt{3600}=\pm 60$; therefore $x=120\pm 60=180$ or 60, both which values answer the conditions of the question. The distance $\therefore$ of C from D is 180 or 60 miles.

7. An oblong pond was surrounded by a terrace walk 4 yards broad; the pond measured 15200 square yards, and the walk 2104 square yards. Required the length and breadth of the pond.

Let $x=$ the length of the pond; then $\frac{15200}{x}=$ the breadth. Hence $(x+4)\times(\frac{15200}{x}+8)=(15200+2104)=17304$; that is, $8x+\frac{121600}{x}+15264=17304$, or $8x^2-2040x=+121600$. Divide by 8, $x^2-255x=-15200$; by case 4, $x=127\cdot5\pm 32\cdot5=160$, or 95, both of which values answer the conditions of the question. The length of the pond, therefore, is 160 or 95 yards, and the breadth $\frac{15200}{160}=95$, or $\frac{15200}{95}=160$ yards.

8. A and B gained \$140 by trade. A's money was three months in trade, and his gain was \$60 less than his stock; and B's money, which was \$50 more than A's, was in trade 5 months. What was A's stock?

Put $140=g$, $60=a$, $50=b$, and $x=$A's stock; then will $x-a$ be A's gain; also, $x+b$ will be B's stock, and $g-(x-a)=(g+a)-x$ B's gain; but the gains must be proportional to the stocks multiplied by the respective times; therefore $3x:5(x+b)::x-a:g+a-x$, or $3x(g+a-x)=(5x+4b)(x-a)$; $\therefore$ $8x^2-8ax-3gx+5bx=5ab$, by multiplying and transposing; or $8x^2-$

$650x=15000$, in numbers by case 2d, $x^2-\frac{325}{4}x+(\frac{325}{8})^2=1875+(\frac{325}{8})^2=\frac{225625}{64}$; by completing the square, its root is $x-\frac{325}{8}=\surd(\frac{225625}{64})=\frac{475}{8}$, and $x=\frac{325}{8}+\frac{475}{8}=\frac{800}{8}=100$.

9 . A tailor bought a piece of cloth for \$147, from which he cut off 12 yards for his own use, and sold the remainder for \$$\frac{481}{8}$ and gained \$$\frac{1}{4}$ per yard. How many yards were there, and what did it cost him per yard?

Let x, $\frac{147}{x}$, and $\frac{481}{4(x-12)}$, $\therefore \frac{147}{x}=\frac{481}{4(x-12)}-\frac{1}{4}$ denote the number of yards, the buying price per yard, the selling price, and the equation, respectively, and $x^2+95x+(\frac{95}{2})^2=7056+\frac{9025}{4}=\frac{37249}{4}$, and $x+\frac{95}{2}=\frac{193}{2}$, or $x=49$, Ans.

10. A farmer received \$288 for a certain quantity of wheat, and an equal sum at a price less by \$3 per bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there? Ans. 32 bushels of wheat, and 48 of barley.

Let x, $x+16$, $\frac{288}{x}$, and $\frac{288}{x+16}$ denote the number and price of a bushel of wheat and barley, respectively, by the question, $\frac{288}{x}=\frac{288}{x+16}+3$, or $\frac{96}{x}=\frac{96}{x+16}+1$, $96(x+16)=96x+x^2+16x$.

General solution by which all such questions may be solved; let $x=$ the number of bushels of wheat; $\therefore x+a=$ the number of barley. Also $\frac{b}{x}=$ the price of a bushel of wheat, and $\frac{b}{x+a}=$ the price of a bushel of barley; $\therefore \frac{b}{x}=\frac{b}{x+a}+c$; $\therefore bx+ab=bx+cx^2+acx$, or $cx^2+acx=ab$. By case first, $x^2+ax=\frac{ab}{c}$, or $x^2+ax+\frac{a^2}{4}=\frac{ab}{c}+\frac{a^2}{4}=\frac{4ab+ca^2}{4c}$; $x+\frac{a}{2}=\surd(\frac{4ab+cx^2}{4c})$, and $x=\surd(\frac{4ab+ca^2}{4c})-\frac{a}{2}$. If $a=16$, $b=288$, and $c=3$, we have $x=\frac{80}{2}=40-8=32$; there were 31 bushels of wheat, and 48 of barley.

11. A and B were dispatched at the same time to a place 90 miles distant, the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him; at what rate did each travel per hour? Ans. A went 10, and B 9 miles per hour.

Let $x=$ the number of miles A went per hour; $\therefore x-1=$ the number B went, and $\frac{90}{x}=\frac{90}{x-1}-1$, or $90(x-1)=90x-x^2+x$, or $x^2-x=90$; by case 2d, $x=10$, the miles.

12. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. How many miles did he go per hour? Ans. 7 miles.

Let $x=$ the number; $\therefore \frac{105}{x}=\frac{105}{x-2}-6$, or $x^2-2x=35$; by case 2d, $x=\pm\sqrt{(35+1)}+1=7$, and $\frac{105}{7}=\frac{105}{5}-6=15$ hours.

13. A and B distributed 1200 dollars each among a certain number of persons; A relieves 40 persons more than B, and B gives 5 dollars a piece to each person more than A. How many persons were relieved by A and B respectively?

Let $x=$ the number of persons B relieves, then $x+40$ will $=$ the number A relieves. But $\frac{1200}{x+40}+5=\frac{1200}{x}\times$ by (x^2+40x), or $1200x+5x^2+200x=1200x+48000$; transpose and divide by 5, I have $x^2+40x=9600$; by case 1, $x=\pm100-20=80$, or 120.

14. A person bought a certain number of sheep for 1140 dollars. Having lost 8 of them, and sold the remainder at 8 dollars a head profit, he is no loser by the bargain. How many sheep did he buy? Ans. 38.

Let x denote the number of sheep; then each sheep will cost $\frac{1140}{x}$. But $\frac{1140}{x-8}=\frac{1140}{x}+8\times(x^2-8x)$, or $x^2-8x=1140$; by case 2, $x=4+\sqrt{1156}=34+4=38$, Ans.

15. A and B set off at the same time to a place at the distance of 300 miles. A travels at the rate of one mile an hour faster than B, and arrives at his journey's end 10 hours before him; at what rate did each person travel per hour?

Let $x=$ the rate per hour B travels; then $x+1=$ the rate A travels. But $\frac{300}{x+1}=\frac{300}{x}-10\times(x^2+x)$, or $300x=300x-10x^2-10x+300$, or $x^2+x=30$, and by case 1, $x=\frac{11}{2}-\frac{1}{2}=\frac{10}{2}=5$, rate B travels per hour, and $5+1=6=$ A's rate per hour.

16. A person bought some sheep for 72 dollars, and found that if he had bought 6 more for the same money, he would have paid 1 dollar less for each; how many did he buy, and what was the price of each? Ans. 18 sheep, and the price 4 dollars.

Let $x, \frac{72}{x}, \frac{72}{x+6}$, and $\therefore \frac{72}{x+6}+1=\frac{72}{x}$, denote the number of sheep bought, the price of one in dollars, the price of one, if he had bought 6 more, and the equation respectively; then $x^2+6x=432$, and by case 1, $x=\pm21-3=18$.

72. *A new method of resolving Equations of all orders, however complicated.*

A universal method, which may be applied to equations of all dimensions, was first given by W. Emerson, in his Algebra in 1764, first edition, with a new demonstration, thus: let $x^2+ax=n$ be the general form of a quadratic equation, let r denote the first figure of the root, and y the remaining part of the root; that is, if the root be integral, then r denotes the left integer, with as many ciphers as figures in the root. If the root consist of an integer and decimals, then r will denote the integer, and y the decimal part; then we shall have $x = r+y$, or $(y+r)$, and then, by substituting r for x in the equation, we have $r^2+ar=n$. Hence, $\frac{n}{r+a}$ will give r, or the first figure of the root. To find the remaining figures, substitute $r+y$ for x in the given equation;

Then $\left.\begin{array}{r} y^2+2ry+\ r^2 \\ ay+ar \end{array}\right\}=n$

$\overline{y^2+a'y+p=n}$, or $y^2+a'y=n-p = n'$. From this equation, which is similar to the first y, can be found $y = s+z$, or $(z+s)$; then by substitution, 2dly,

$\left.\begin{array}{r} z^2+\ 2sz+s^2 \\ s^2+as \end{array}\right\}= n.$

$\overline{z^2+a'z+p}\ =n$, or $z^2+a''z=n'-p=n''$. Also, $\frac{n'}{s+a}=s'$

From this equation z can be determined. Let $z = t+q$; then $\frac{n'}{t+a}=t$. In the same manner, we find $\frac{m'''}{u+a''},,=u$, &c.

Now, the value a', a'', a''', &c. can be easily determined, and hence we have the following expressions for s, t, u, &c. or the remaining figures of the root, thus

$$\frac{n}{r+a}=r;\ \frac{n'}{s+a'}=\frac{n'}{s+2r+a}=s;\ \frac{n''}{t+a''}=\frac{n''}{t+2s+2r+a}=t;$$

$$\text{and } \frac{n'''}{u+a'''}=\frac{n''}{u+2t+2s+2r+a}=u, \text{ \&c.}$$

I shall close the subject of quadratic equations by one observations. Consequently this method appears to be a generalization of the extraction of the square root, if we suppose $a=0$, then $x^2=n$, or $x=\sqrt{n}$, and the formulas become

$$\frac{n}{r}=r,\ \frac{n'}{s+2r}=s,\ \frac{n''}{t+2s+2r}=t, \text{ and } \frac{n'''}{u+2t+2s+2r}=u,$$

This is a demonstration of the rule usually given for the extraction of the square root of numbers.

On Ratios, Proportion, and Variation.

73. By Ratio is meant the relation which one quantity bears to another, with respect to magnitude. It is evident that this relation can exist only between quantities of a similar kind; thus, a number must be compared with a number, a line with a line, &c. &c.; and it would be absurd to compare a certain number of feet with a certain number of dollars, &c. &c.

74. There are two ways in which the magnitude of quantities may be compared. In the first place, they may be compared with regard to their difference; and then the question asked is, "How much one quantity is greater or less than another." The relation which quantities bear to each other in this respect, is called their arithmetical ratio. The other way in which they may be compared, is, by inquiring, "How often one quantity is contained in the other." This relation between quantities is called their geometrical ratio. The term ratio, when simply applied, is generally understood in the latter sense, and it is in this sense that the word will be made use of in the present chapter.

75. In considering how often one quantity is contained in another, the natural process is to divide the one by the other. Thus, in comparing the number 12 with the numbers 4 and 3, we know that 4 is contained in 12 three times, and that 3 is contained in the same number four times; from which we infer, that the ratio of 12 : 3* is greater than the ratio of 12 : 4, the magnitude of a ratio being measured by the number of times one quantity is contained in another. For the same reason, the ratio of 11 : 7 is said to be less than the ratio of 11 : 5. When the ratio is thus expressed, the first term of it is called the antecedent, the last term the consequent, of that ratio.

76. From this mode of estimating the magnitude of a ratio, it appears that when the consequent of a ratio is not an aliquot part of the antecedent, the value of the ratio must be expressed by a fraction whose numerator is the antecedent, and denominator the consequent, of that ratio. Thus, the magnitude of the ratio of 15 : 7 is expressed by the fraction $\frac{15}{7}$, and of the ratio 4 : 13, by the fraction $\frac{4}{13}$. When the antecedent of a ratio is greater than the consequent, it is called a ratio of greater inequality; when the antecedent is less than the consequent, a ratio of lesser inequality;

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* In expressing the ratio of two quantities, the word *to* is generally supplied by two dots; thus, the ratio of a to b is expressed by $a : b$.

and if the two terms of a ratio be the same, then it is said to be a ratio of equality.

77. The foregoing definitions evidently apply only to those instances in which the consequent of a ratio is contained a certain number of times in the antecedent, or to those in which the magnitude of the ratio may be expressed by some definite fraction. It does not, therefore, comprehend such ratios as $\sqrt{2} : 5$; $\sqrt{3} : \sqrt[3]{7}$; $4 : \sqrt[4]{3}$, &c., where the values of the quantities $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$, $\sqrt[4]{3}$, &c., can only be expressed in decimal fractions which do not terminate. The ratio which exists between quantities of this latter kind, when the radical quantity is expressed by a decimal fraction, is called their approximate ratio.

78. Proportion consists in the inequality of ratios; thus, since 4 is contained in 12 the same number of times that 6 is in 18, the ratio of $12 : 4$ is said to be equal to the ratio of $18 : 6$, or, in other words, that $12 : 4 :: 18 : 6$.* Of the four terms of which every proportion consists, the first and last terms are called the extremes, and the second and third the means, of that proportion.

79. If there be a set of quantities related together in the following manner, viz. $a : b :: b : c :: c : d :: d : e$, &c., where the consequent of every preceding ratio is the antecedent of the following one, then the quantities a, b, c, d, e, &c. are said to be in continued proportion; and if only three quantities be concerned, as in the proportion $a : b :: b : c$, then b is said to be a mean proportional between the two extremes a and c.

80. Since the proportion $a : b :: c : d$ expresses the equality of the ratios $a : b$ and $c : d$, and since the magnitude of the ratio $a : b$ is measured by the fraction $\frac{a}{b}$, and that of the ratio $c : d$ by the fraction $\frac{c}{d}$, it follows that $\frac{a}{b} = \frac{c}{d}$, or that when four quantities are proportional, the quotient of the first divided by the second is equal to the quotient of the third divided by the fourth; and *vice versa,* if there be four quantities, a, b, c, d, such, that $\frac{a}{b} = \frac{c}{d}$, then those four quantities are proportional, or $a : b :: c : d$.

On the Comparison and Composition of Ratios.

81. *On the comparison of ratios.*

I. Since the ratio of $a : b$ may be expressed by the fraction

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* In stating a proportion, the words *is to* and *to* are generally supplied by two dots, and the word *as* by four dots; thus, the proportion a is to b as c to d, is expressed by $a : b :: c : d$.

$\frac{a}{b}$, let the numerator and denominator of this fraction be multiplied by any quantity m, (m being either integral or fractional,) then $\frac{ma}{mb}=\frac{a}{b}$, and therefore the ratio of $ma : mb$ is the same with the ratio of $a : b$; from which we infer, that if the terms of a ratio be multiplied or divided by the same quantity, it does not alter the value of the ratio. Hence also it appears, that a ratio is reduced to its lowest terms by dividing its antecedent and consequent by their greatest common measure.

II. Ratios are compared together by reducing the fractions by which their values are respectively represented, to a common denominator. Thus, the ratio of $8 : 5$ is represented by the fraction $\frac{8}{5}$, and the ratio of $9 : 6$ by the fraction $\frac{9}{6}$; reduce these fractions to others of the same value, having a common denominator, and they become $\frac{48}{30}$ and $\frac{45}{30}$ respectively; and since $\frac{48}{30}$ is greater than $\frac{45}{30}$, the ratio $8 : 5$ is greater than the ratio of $9 : 6$.

III. A ratio of greater inequality is diminished, and a ratio of lesser inequality is increased, by adding the same quantity to both its terms. Let $a+b : a$ represent a ratio of greater inequality, and let x be added to each of its terms, and it becomes the ratio of $a+b+x : a+x$. Now the ratio of $a+b : a = \frac{a+b}{a}$, and that of $a+b+x : a+x=\frac{a+b+x}{a+x}$; let these fractions be reduced to others of the same value, having a common denominator, and they become $\frac{a^2+ab+ax+bx}{a(a+x)}$ and $\frac{a^2+ab+ax}{a(a+x)}$, respectively; and since $a^2+ab+ax+bx$ is evidently greater than $a^2+ab+ax$, the ratio of $a+b : a$ is greater than the ratio of $a+b+x : a+x$; i. e. the ratio of $a+b : a$ has been diminished by adding x to each of its terms. Next, let $a-b : a$ represent a ratio of lesser inequality; then, proceeding with the fractions $\frac{a-b}{a}$ and $\frac{a-b+x}{a+x}$, as in the former instance, the resulting fractions are $\frac{a^2-ab+ax-bx}{a(a+x)}$ and $\frac{a^2-ab+ax}{a(a+x)}$; and since $a^2-ab+ax-bx$ is less than $a^2-ab+ax$, the ratio of $a-b : a$ is less than the ratio of $a-b+x : a+x$, and consequently the ratio of $a-b : a$ has been increased by adding x to each of its terms. In the same manner it might be shown that a ratio of greater inequality is increased, and a ratio of lesser inequality is diminished, by subtracting the same quantity from each of its terms.

82. *On the composition of ratios.*

I. Ratios are compounded together by multiplying their antecedents together for a new antecedent, and their consequents together for a new consequent. Thus, if the ratio of $a : b$ be compounded with the ratio of $c : d$, the resulting ratio is that of $ac : bd$; or if the ratios $4 : 3$, $5 : 2$, and $7 : 1$, be compounded together, there results the ratio of $4\times5\times7 : 3\times2\times1$, or of $140 : 6$, or (dividing each term by 2) of $70 : 3$.

II. If the same ratio be compounded with itself once, twice, thrice, &c., the resulting ratios are those of $a^2 : b^2$, $a^3 : b^3$, $a^4 : b^4$; &c. &c. The ratio of $a^2 : b^2$ is called the duplicate ratio of $a : b$; $a^3 : b^3$, the triplicate; $a^4 : b^4$, the quadruplicate, &c. &c.; and as these ratios receive their denominations from the indices of the several powers of a and b, the ratio of $\sqrt{a} : \sqrt{b}$ is called the subduplicate ratio of $a : b$; the ratio of $\sqrt[3]{a} : \sqrt[3]{b}$, the subtriplicate, &c. &c.

III. If a set of ratios, whereof the consequent of the preceding ratio is the same with the antecedent of the succeeding one, be compounded together, the resulting ratio is that of the first antecedent to the last consequent. Thus, when the ratios of $a : b$, $b : c$, $c : d$, $d : e$, are compounded together, the resulting ratio is that of $abcd : bcde$, or, (dividing by bcd) that of $a : e$, or of the first antecedent : the last consequent; and the same will be the case whatever be the number of ratios.

IV. A ratio of greater inequality compounded with another ratio, increases it; and a ratio of lesser inequality compounded with another ratio, diminishes it. Thus, let $1+n : 1$ represent a ratio of greater inequality, and let it be compounded with the ratio $a : b$, the resulting ratio is that of $a + na : b$, which is evidently greater than the ratio of $a : b$. On the other hand, let $1 - n : 1$ represent a ratio of lesser inequality, and let it be compounded with the ratio of $a : b$, then the resulting ratio is that of $a - na : b$, which is evidently less than the ratio of $a : b$.

1. Reduce the ratio of $360 : 315$, and $1595 : 667$, to their lowest terms.

$360 : 316, = \dfrac{360}{315} \div 45$ the greatest common measure $\dfrac{8}{7} = 8 : 7$,

and $1595 : 667, = \dfrac{1595}{667} \div 29$, greatest common measure, $= \dfrac{55}{33}$ $= 55 : 23$.

2. Reduce the ratio $a^3+2a^2x : a^2$ to its lowest terms.

Let $a^3+2a^2x : a^2 = \dfrac{a^3+2a^2x}{a^2} \div a^2 = \dfrac{a+2x}{1}$. Hence $a+2x : 1$.

3. Which is the greater, the ratio of 16 : 15, or that of 17 : 14?
$\frac{16}{15}\times 14=\frac{227}{210}$; $\frac{17}{14}\times 15=\frac{255}{210}$ $\therefore$ is the greatest.

4. Which is the least of the three ratios, 20 : 17, 22 : 18, or 25 : 23? and which is the greatest of the three ratios, 8 : 7, 6 : 5, and 10 : 9?
$\frac{20}{17}$; $\frac{22}{18}$; $\frac{25}{23}$ $\times$ by the least common multiple of the denominators, and $\frac{20}{17}=4140 : 3519$; $\frac{22}{18}4301 : 3519$; $\frac{25}{23}=3825:3519$ least, and $\frac{8}{7}=360 : 315$; $\frac{6}{5}=378 : 315$ $\therefore$ greatest, $\frac{10}{9}=350 : 315$.

5. Which is the greater, the ratio of $a+2 : \frac{1}{2}a+4$, or that of $a+4 : \frac{1}{2}a+5$? Ans. the ratio of $a+4 : \frac{1}{2}a+5$.
$\frac{a+2}{\frac{1}{2}a+4}$ and $\frac{a+4}{\frac{1}{2}a+5}$ $\times$ by the alternate denominator, and $\frac{a+2}{\frac{1}{2}a+4}$ $=\frac{a^2}{3}+6a+16 : \frac{a^2}{4}+\frac{9a}{4}+20$. Hence $a+4 : \frac{1}{2}a+5$ the greatest.

6. Compound together the ratios of 11 : 3, 7 : 2, and 5 : 9.

$$\frac{11\times 7\times 5}{3\times 2\times 9}=\frac{385}{54}=385 : 54.$$

7. Compound together the ratios of 15 : 12, 6 : 7, and 9 : 4; and then reduce the resulting ratio to its lowest terms.
$\frac{15\times 6\times 9}{12\times 7\times 4}=\frac{810}{336}\div 6$ common measure, $=\frac{135}{56}=135 : 56$.

8. Express in the simplest terms the ratio compounded of $a^2-x^2 : a^2$, $a+x : b$, and $b : a-x$.
$\frac{(a^2-x^2)\times(a+x)\times b}{a^2\times b\times(a-x)}=\frac{(a+x)\times(a+x)}{a^2}=(a+x)^2 : a^2$, by cancelling those quantities which are common to the terms.

9. If the ratios of $x+y : a$, $x-y : b$, and $b : \frac{x^2-y^2}{a}$ be compounded together, show that the resulting ratio is a ratio of equality.

$$\frac{(x+y)\times(x-y)\times b}{a\times b}\times\frac{a}{x^2-y^2}=\frac{x^2-y^2}{x^2-y^2}=1 : 1.$$

10. If the ratios of $3a+2 : 6a+1$, and of $2a+3 : a+2$ be compounded together, is the resulting ratio a ratio of greater or lesser inequality?

$\frac{(3a+2)\times(2a+3)}{(6a+1)\times(a+2)}=\frac{6a^2+13a+6}{6a^2+13a+2}=6a^2+13a+6 : 6a^2+13a-$

$\therefore$ a ratio of greater inequality.

11. What are the least numbers in the ratio compounded the three following ratios, viz. the ratio of 7 : 5, the duplicate tio of 4 : 9, and the triplicate ratio of 3 : 2?

$$\frac{7\times4\times3}{5\times9\times2}=\frac{14}{15}, \therefore 14 \text{ and } 15.$$

12. Compound the subduplicate ratio of $x^2 : y^2$ with the qu ruplicate ratio of $\sqrt{x} : \sqrt{y}$.

$\frac{x}{y}$ = subduplicate ratio of $x^2 : y^2$, and $\frac{x^2}{z^2}$ = quadruplicate ra of $\sqrt{x} : \sqrt{y}$; $\therefore \frac{x\times x^2}{y\times y^2}=x^3 : y^3$.

On Proportion.

83. The most useful Theorems relating to proportional qu tities are the following:

Th. 1. If four quantities be proportional, the product of extremes will be equal to the product of the means; for $a : b :: c : d$, then, by Art. 80, $\frac{a}{b}=\frac{c}{d}$, $\therefore ad=bc$. Hence, al it follows, that if any three terms of a proportion be known, fourth may be found; for, from the equation $ad = bc$, we ha $a=\frac{bc}{d}$; $b=\frac{ad}{c}$; $c=\frac{ad}{b}$; and $d=\frac{bc}{a}$.

Th. 2. The converse of the foregoing theorem is also tru viz.: If the product of any two quantities be equal to the prod of two others, those four quantities will constitute a proporti provided that the terms of one product be made the means, a the terms of the other product be made the extremes, of such p portion. Thus, if the four quantities a, b, c, d, be such t $ad=bc$, then (dividing by bd) $\frac{a}{b}=\frac{c}{d}$; $\therefore$ by Art. 80, $a : b :: c :$

Th. 3. If three quantities be proportional, the product of two extremes is equal to the square of the mean; for, $a : b :: b : c$, then, by theorem 1, $ac=b^2$. Hence also it follo that a mean proportional between any two quantities is equal the square root of their product; for, let x be a mean prop tional between a and c, then $a : x :: x : c$; $\therefore x^2 = ac$, a $x = \sqrt{ac}$.

Th. 4. If four quantities be proportional, they will also be proportional when taken inversely or alternately; thus, if $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$; invert the fractions, then $\frac{b}{a}=\frac{d}{c}$; $\therefore$ $b : a :: d : c$. Again, since $ad=bc$, then (dividing by cd) we have

$$\frac{ad}{cd}=\frac{bc}{cd}, \text{ or } \frac{a}{c}=\frac{b}{d}; \therefore a : c :: b : d.$$

Th. 5. If there be six proportional quantities, and the first be to the second as the third to the fourth, and the third to the fourth as the fifth to the sixth, then will the first be to the second as the fifth to the sixth. For let $a : b :: c : d$, and $c : d :: e : f$: then $\frac{a}{b}=\frac{c}{d}$; and $\frac{c}{d}=\frac{e}{f}$; $\therefore$ $\frac{a}{b}=\frac{e}{f}$, or, by Art. 80, $a : b :: e : f$.

Th. 6. If four quantities be proportional, then the sum or difference of the first and second will be to the second as the sum or difference of the third and fourth is to the fourth. For let $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$; add 1 to, or subtract it from, each side of the equation, then $\frac{a}{b}\pm 1=\frac{c}{d}\pm 1$; $\therefore$ $\frac{a\pm b}{b}=\frac{c\pm d}{d}$ consequently, by Art. 80, $a\pm b : b :: c\pm d : d$.

Th. 7. If four quantities be proportional, the first is to the sum or difference of the first and second as the third to the sum or difference of the third and fourth. For, by theorem 6, $a\pm b : b :: c\pm d \,.\, d$, and alternately $a\pm b : c\pm d :: b : d$; but, by theorem 4, $b : d :: a : c$; hence, by theorem 5, $a\pm b : c\pm d :: a : c$, and alternately $a\pm b : a :: c\pm d : c$; $\therefore$ inversely $a : a\pm b :: c : c\pm d$.

Th. 8. If four quantities be proportional, then the sum of the first and second is to their difference as the sum of the third and fourth is to their difference. For, by theorem 6, $\frac{a+b}{b}=\frac{c+d}{d}$; and $\frac{a-b}{b}=\frac{b-d}{d}$; invert the last two fractions, then $\frac{b}{a-b}=\frac{d}{c-d}$ hence $\frac{a+b}{b}\times\frac{b}{a-b}=\frac{c+d}{d}\times\frac{d}{c-d}$, or $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; $\therefore$ by Art. 80, $a+b : a-b :: c+d : c-d$.

Th. 9. If four quantities be proportional, and any equimultiples or equal parts whatever be taken of the first and second, and also of the third and fourth, then will the resulting quantities taken in the same order, be still proportional. For let $a : b :: c :$

d; then, by Case I, Art. 82, the ratio of $ma : mb$ is the same with the ratio of $a : b$; and for the same reason, the ratio of $nc : nd$ is the same with the ratio of $c : d$; hence, (Article 78,) $ma : mb :: nc : nd$, where m and n may be any quantities whatever, either integral or fractional.

TH. 10. The same theorem is true, if any equimultiples or equal parts whatever be taken of the first and third, and also of the second and fourth; for since $\frac{a}{b}=\frac{c}{d}$, multiply each side of the equation by $\frac{m}{n}$, then $\frac{ma}{nb}=\frac{mc}{nd}$; $\therefore ma : nb :: mc : nd$, where m and n may be any quantities whatever, either integral or fractional.

TH. 11. If four quantities be proportional, any powers or roots of those quantities will also be proportional. For since $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a^n}{b^n}=\frac{c^n}{d^n}$; $\therefore a^n : b^n :: c^n : d^n$, where n may be any number, either integral or fractional.

TH. 12. If the corresponding terms of two sets of proportionals be multiplied together, or divided by each other, the resulting quantities, taken in order, will still be proportional. Thus, let

$$\left.\begin{array}{c} a : b :: c : d \\ \text{and} \\ e : f :: g : h \end{array}\right\} \left.\begin{array}{c} \text{then } \frac{a}{b}=\frac{c}{d} \\ \\ \text{and } \frac{e}{f}=\frac{g}{h} \end{array}\right\} \text{ hence } \frac{ae}{bf}=\frac{cg}{dh}, \text{ or } ae : bf :: cg : dh.$$

Again, by theorem 1, $ad=bc$, and $eh=fg$; $\therefore \frac{ad}{eh}=\frac{bc}{fg}$; hence by theorem 2, $\frac{a}{e}:\frac{b}{f}::\frac{c}{g}:\frac{d}{h}$. The same will evidently be true of any number of proportions.

TH. 13. If there be two rows of proportional quantities, wherein the second and fourth terms of the first row are the same with the first and third terms of the second row, then will the remaining quantities, taken in order, be proportional. Thus,

let $a : b :: c : d$,
and $b : e :: d : f$; then, by theorem 12, $ab : be :: cd : df$,
or, (reducing each ratio to its lowest terms,) $a : e :: c : f$.

TH. 14. If there be a set of proportional quantities, $a : b :: c : d :: e : f :: g : h$, &c., then will the first be to the second as the sum of all the antecedents to the sum of all the consequents. For, since $ab=ba$, and (by theorems 1 and 5,) $ad = bc$,

$af=be$, $ah=bg$, &c., we have $ab+ad+af+ah+$&c.$=ba+bc+be+bg+$&c., or $a(b+d+f+h+\text{&c.})=b(a+c+e+g+\text{&c.})$ $\therefore$ (by Th. 2,) $a : b :: a+c+e+g+\text{&c.} : b+d+f+h+\text{&c.}$

Th. 15. If $a : b :: b : c :: c : d :: d : e$ &c., as in article 79, then $a : c :: a^2 : b^2$, or in the duplicate ratio of $a : b$;
$a : d :: a^3 : b^3$, or in the triplicate ratio of $a : b$;
$a : e :: a^4 : b^4$, or in the quadruplicate ratio of $a : b$;
&c., &c., &c.

$$\left.\begin{array}{r}\text{For } a : b :: a : b;\\ \text{and } b : c :: a : b;\\ \therefore \text{ by Th. 12, } a : c :: a^2 : b^2.\end{array}\right\} \qquad \begin{array}{r}\text{Again, } a : c :: a^2 : b^2;\\ \text{but } c : d :: a : b;\\ \therefore \text{ Th. 12, } a : d :: a^3 : b^3.\end{array}$$

Moreover, $a : d :: a^3 : b^3$
but $d : e :: a : b$;
$\therefore$ by Th. 12, $a : e :: a^4 : b^4$, &c. &c. &c.

Arithmetical Progression is when a series of numbers or quantities increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, &c. and a, $a+d$, $a+2d$, $a+3d$, &c., are increasing series in arithmetical progression, the common differences of which are 2 and d.

And 15, 12, 9, 6, &c. and a, $a-d$, $a-2d$, $a-3d$, &c. are decreasing series in arithmetical progression, the common differences of which are 3 and d.

The most useful properties of arithmetical proportion and progression are contained in the following theorems:

1. If four quantities are in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, if the two proportionals be 2, 5, 7, 10, or a, b, c, d; then will $2+10=5+7$, and $a+d=b+c$.

2. And if three quantities be in arithmetical proportion, the sum of the two extremes will be double the mean.

Thus, if the proportionals be 3, 6, 9, or a, b, c, then will $3+9=2\times6=12$, and $a+c=2b$.

3. Hence, an arithmetical mean between any two quantities is equal to half the sum of those quantities.

Thus, an arithmetical mean between 2 and 4 is $=\frac{2+4}{2}=3$; and between 5 and 6, it is $=\frac{5+6}{2}=5\frac{1}{2}$.

And an arithmetical mean between a and b is $\frac{a+b}{2}$.

4. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, when the number of terms is odd.

Thus, if the series be 2, 4, 6, 8, 10, then will $2+10=4+8=2\times6=12$.

If two or more arithmetical means between any two quantities be required, they may be expressed as below.

Thus, $\frac{2a+b}{3}$ and $\frac{a+2b}{3}$ = two arithmetical means between a and b, a being the less extreme and b the greater.

And $\frac{na+b}{n+1}$, $\frac{(n-1)a+2b}{n+1}$, $\frac{(n-2)a+3b}{n+1}$, &c. to $\frac{a+nb}{n+1}$ = any number (n) of arithmetical means between a and b; where $\frac{b-a}{n+1}$ is the common difference; which being added to a, gives the first of these means; and then to this last, gives the second; and so on.

And, if this series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, then will $a+(a+4d)=+d+(a+3d)=2(a+2d)$.

5. The last term of any increasing arithmetical series is equal to the first term plus the product of the common difference by the number of terms less one; and if the series be decreasing, it is equal to the first term minus that product.

Thus, the last term (l) of the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c. continued to n terms, is $l=a+(n-1)d$.

And the last term of the series a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c. to n terms, is $l=a-(n-1)d$.

6. The sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms. Thus, the sum of the series 2, 4, 6, 8, 10, 12, is $=(2+12)+\frac{6}{2}=14\times3=42$.

And if the series be $a+(a+)+(a+2d)+(a+3d)+$, &c. to n terms, we shall have $s=(a+l)\times\frac{1}{2}n$, where l is the last term, n the number of terms, and s the sum of the series.

Or, the sum of any increasing arithmetical series may be found by adding the product of the common difference by the number of terms less one, to twice the first term, and then multiplying the result by half the number of terms.

And, if the series be decreasing, its sum will be found by subtracting the product of the common difference by the number of terms less one, from twice the first term, and then mnltiplying the result by half the number of terms, as before.*

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* The sum of any number of terms (n) of the series of natural numbers 1, 2, 3, 4, 5, 6, 7, &c. is $=\frac{n(n+1)}{4}$. Thus, $1+2+3+4+5+$, &c. continued to 100 terms, is $=\frac{100\times101}{2}=50\times101=$ 5050.

Thus, if the series be $a+(a+d)+(a+2d)+(a+3d)+(a+4d)+$, &c. continued to n terms, we shall have $s=\{2a+(n-1)d\}\times\frac{n}{2}$. And, if the series be $a+(a-d)+(a-2d)+(a-3d)+(a-4d)+$, &c. continued to n terms, we shall have

$$s=\{2a-(n-1)d\}\times\frac{n}{2}.$$

Case I. The sum of a series of quantities in arithmetical progression is found by multiplying the sum of the first and last terms by half the number of terms.

Let $a =$ first term, $d =$ common difference,
$n =$ number of terms, and $s =$ sum of the series.

$$\begin{array}{l} a + a+d + a+2d + a+3d \text{ \&c.}\ldots\ldots\ldots a+(n-1)d = s \\ a+(n-1)d + a+(n-2)d + a+(n-3)d +, \text{ \&c.}\ldots.a = s \\ \hline \text{By addition } 2a+(n-1)d + 2a+(n-1)d + \text{\&c. to } n \text{ terms} = 2s \end{array}$$

$$\therefore \{2a+(n-1)d\} \,.\, n = 2s.$$

Whence $s=\{2a+(n-1)d\}.\frac{n}{2}=\{a+a+(n-1)d\}\frac{n}{2}$

$$=\{\text{first term} + \text{last term}\}.\frac{\text{No. of terms}}{2}.$$

Cor. 3. Any three of the four quantities s, a, n, d, being known, the fourth may be found.

If s, d, n be given, ——— $a=\frac{2s-dn(n+1)}{2n}$.

$s=\{2a+(n-1)d\}\frac{n}{2}$; by actual multiplication, $2an+dn^2-nd=2s$, or $2an=2s-dn^2+dn$; $\therefore a=\frac{2s-n^2d+dn}{2n}$.

If a, s, n be given, we find d $\therefore d=\frac{2}{n}.\frac{s-an}{n-1}$.

$bn^2-bn=2s-2an$, or $(n^2-n)d=2s-2an$; $\therefore d=\frac{2s-2an}{n^2-n.}$.

If s, d, a be given ——— $n=\frac{\sqrt{\{(2a-d)^2+8sd\}}-(2a-d)}{2d}$.

To find n, we have, by transposition, $dn^2+2an-dn=2s$, or $dn^2+(2a-d)n=2s$; $\therefore n^2+\frac{2a-d}{d}\times n=\frac{2s}{d}$; and, by case 1, quadratics, we have $n=\frac{d-2a}{2d}\pm\frac{1}{2d}\sqrt{\{(2a-d)^2+8ds\}}$.

Cor. 2. If the series be a decreasing one, d is negative, and $s=\{2a-(n-1)d\}\cdot\frac{n}{2}$.

1. The first term of an increasing arithmetical series is 3, the common difference 2, and the number of terms 20. Required the sum of the series.

First, $3+2(20-1)=3+2\times19=3+38=41$, the last term, and $(3+41(\times\frac{20}{2}=44\times\frac{20}{2}=44\times10=440$, the sum. Or, $\{2\times2+2(20-1)\times\frac{20}{2}=(6+2\times19)\times10=(6+38)\times10=44\times10=440$, as before.

2. The first term of an increasing arithmetical series is 100, the common difference 3, and the number of terms 34. Required the sum of the series.

First, $100-3(34-1)=100-3\times33=100-99=1$, the last term, and $(100+1)\times\frac{34}{2}=101\times\frac{34}{2}=101\times17=1717$, the sum required. Or, $\{2\times100-3(34-1\})\times\frac{34}{2}=(200-3\times33)\times17=(200-99)\times17=101\times17=1717=$ sum, as before.

3. Required the sum of the odd numbers 1, 3, 5, 7, 9, &c. continued to 101 terms.

Here the formula $s=\{2a+(n-1)a\}\frac{n}{2}$ becomes $\{2+100\times2\}\times\frac{101}{2}=202\times\frac{101}{2}=101^2=10201$.

4. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in the compass of a day?

Here $s=(a+l)\frac{n}{2}=(1+24)\times12=300$, Ans.

5. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21. Required the sum of the series. Ans. 140.

Here the formula, $l=a+(n-1)d$, becomes $l=2+(365-1)2=2+728=730$, Ans.

6. One hundred apples being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel, who shall bring them, one by one, to a basket placed at the distance of a yard from the first apple?

Here the 1st term is $1+1=2$, and the last $100+100=200$, the number of terms 100. Therefore $(a+l)\frac{n}{2}=(2+200)50=10100$ yards, or 5 miles 1300 yards, Ans.

7. Required the 365th term of the series of even numbers 2, 4, 6, 8, 10, 12, &c. Ans. 730.

Also, the sum of any number of terms (n) of the series of odd numbers, 1, 3, 5, 7, 9, 11, &c. is $=n^2$. Thus, $1+3+5+7+9+$, &c., continued to 50 terms, is $=50^2=2500$.

To this we may add, that if any three of the quantities a, d, n, s, be given, the fourth may be found from the equation

$s=\{2a\pm(a-1)d\}\times\frac{n}{2}$, or $(a+l)\times\frac{n}{2}$, where the upper sign $+$ is to be used when the series is increasing, and the lower sign $-$ when it is decreasing; also the last term $l=a\pm(n-1)d$, as above.

Here $s=\{2a-(n-1)d\}\frac{n}{2}=\{20-20\times\frac{1}{3}\}\times\frac{21}{2}=(20-\frac{20}{3})\times\frac{21}{2}=\frac{40}{3}\times\frac{21}{2}=140$, Ans.

The following problems and theorems contain the whole practice of arithmetical progression, with the rules already given.

Given l or a = the first or least term, g = the greatest term, n = the number of terms, s = the sum of the term or series, and b or d = the common excess or difference.

Case 1. Given l, g, and n, to find s and d.

Theo. 1. $(l+g)\frac{n}{2}=s.$ Theo. 2. $\frac{g-l}{n-1}=d.$

Problem 1. Given the least term, the greatest term, and the number of terms, of an arithmetical progression, to find the sum of the terms.

Rule. To the least term add the greatest, multiply the sum by half the number of terms, and the product will be the sum of the terms.

Prob. 2. Given the least term, the greatest term, and the number of terms, to find the common excess or difference.

Rule. Divide the difference between the greatest and the least term by the number of terms less unity, and the quotient will be the common excess, or difference.

Case 2. Given l, g, and s, to find n and d.

Theo. 3. $\frac{2s}{g+l}=n.$ Theo. 4. $\frac{(g+l)\times(g-l)}{2s-(g+l)}=d.$

Case 3. Given l, g, and d, to find n and s.

Theo. 5. $\frac{g-l}{a}+1=n$ Theo. 6. $\frac{(g-l)}{d}+1\times\frac{g+l}{2}=s.$

Prob. 3. Given the least term, the greatest term, and the common excess, or difference, to find the number of terms.

Rule. Divide the difference between the greatest and the least term, by the common excess, or difference, the quotient, increased by an unit, will give the number of terms.

Prob. 4. Given the greatest term, the number of terms, and the common excess, or difference, to find the least term.

Rule. Multiply the common excess, or difference, by the num-

Note 1. If three numbers are in arithmetrical progression, the sum of the extremes will be equal to double the mean; and the product of the extremes, increased by the square of the common difference, will be equal to the square of the mean. Thus, if 5, 7, 9, are in arithmetical progression, then will $5+9=7\times2$, and $(9+5)+(2+2)=7\times7$.

2. If four numbers are in arithmetical progression, the sum of

ber of terms less 1; subtract the product from the greatest term, and the remainder will be the least term.

Case 4. Given l, n, and s, to find g and d.

Theo. 7. $\frac{2s}{n}-l=g$. Theo. 8. $\frac{(s-ln)\times 2}{(n-1)\times n}=\frac{2s-2ln}{n^2-n}=d$.

Case 5. Given l, n, and d, to find g and s.

Theo. 9. $l+(n-1)\times d=g$. Theo. 10. $\{(nd-d)+2l\}\times\frac{n}{2}=s$.

Prob. 5. Given the number of terms, the common excess, or difference, and the sum of the terms, to find the least term.

RULE. Divide the sum of the terms by the number of terms; and, from the quotient subtract half the product of the common excess, or difference, by the number of terms less 1; the remainder will be the least term.

Case 6. Given l, s, and d, to find g and n.

$\frac{1}{2}\sqrt{\{8ds+(2l-d)^2\}}-\frac{1}{2}d=g$. $\frac{\sqrt{\{8ds+(2l-d)^2\}}-(2l-d)}{2d}=n$.

Case 7. Given g, n, and s, to find l and d.

Theo. 13. $\frac{2s}{n}-g=l$. Theo. 14. $\frac{(ng-s)\times 2}{(n-1)\times m}=\frac{2ng-2s}{n^2-n}=d$.

Prob. 6. Given the least term, the number of terms, and the common excess, or difference, to find the greatest term.

RULE. Multiply the number of terms by the common excess, or difference, and to that product add the least term; from this sum subtract the common excess, or difference, and the remainder will be the greatest term.

Case 8. Given g, n, and d, to find l and s.

Theo. 15. $g-(n-1)d=l$. Theo. 16. $\{(2g+d)-nd\}\times\frac{n}{2}=s$.

Case 9. Given g, s, and d, to find l and n.

Theo. 17. $\frac{1}{2}\sqrt{\{(2g+d)^2-8ds\}}+\frac{1}{2}d=l$. 18. See p. 186, cor. 1.

Case 10. Given n, s, and d, to 9nd l and g.

Theo. 19. $\frac{s}{n}-\{(n-1)\times\frac{1}{2}d\}=l$. 20. $\frac{s}{n}+\{(n-1)\times\frac{1}{2}d\}=g$.

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the two extremes will be equal to the sum of the means. Thus, if 2, 5, 8, 11, are in arithmetical progression,

Then will $2+11=5+8$.

3. If a series of numbers (consisting of any number of terms) are in arithmetical progression, the sum of the extremes will always be equal to the sum of any two means equidistant from the extremes; or to double the mean if the terms be odd.

Thus, if 3, 5, 7, 9, 11, 13, &c. are in arithmetical progression,

Geometical Progression.

85 When a series of numbers increase by a common multiplier, or decrease by a common divisor, those numbers are said to be in geometrical progression; such as 2, 4, 8, 16, *&c. &c.* The first and last terms are usually called the extremes, and the common multiplier or divisor the ratio.

Note 1. If three numbers be in geometrical progression, the product of the two extremes will be equal to the square of the mean. Thus, if 3, 9, 27, be in geometrical progression,

Then will $3\times27=9\times9$.

2. If four numbers be in geometical progression, the product of the two extremes will be equal to the product of the means.

Thus, if 2, 4, 8, 16, be in geometrical progression,

Then will $2\times16=4\times8$.

3. If a series of numbers, consisting of any number of terms, be in geometrical progression, the product of the two extremes will be equal to the product of any two means equidistant from the extremes; or to the square of the mean, if the terms be odd.

Thus, if 1, 2, 4, 8, 16, 32, *&c.* be in geometrical progression,

Then will $1\times32=2\times16=4\times8$.

Or, if 1, 2, 4, 8, 16, *&c.* be in geometrical progression,

Then will $1\times16=2\times8=4\times4$.

4. If, out of any series of numbers in geometrical progression, there be taken any series of equidistant terms, that series will likewise be in geometrical progression.

Thus, if 2, 4, 8, 16, 32, 64, *&c.* be in geometrical progression,

Then will 4, 16, 64, *&c.* be in geometrical progression.

Problem I. Given the number of terms the ratio, and either

Then will $3+13=5+11=7+9$. Or, if 1, 4, 7, 10, 13, &c. are in arithmetical progression, then will $1+13=4+10=7\times2$.

4. If, out of any series of numbers in arithmetical progression, there be taken any series of equidistant terms, that series will also be in arithmetical progression.

Thus, if 2, 4, 6, 8, 10, 12, 14, &c. are in arithmetical progression, then will 4, 8, 12, &c. be in arith. prog.

5. In any series of numbers in arithmetical progression the common excess, or difierence, is as often repeated as there are terms in the progression, wanting one; viz. every term, except the first, is continually increased or diminished by the common excess or difference.

6. The rules for an ascending or descending series are the same; for a descending series becomes an ascending one by beginning at the least term.

of the extreme terms, of a limited geometrical series, to find the other extreme.

Rule. Write down a few terms of a geometrical series, beginning with, and formed by the given ratio; over which place the arithmetical series 1, 2, 3, 4, 5, &c. as indices; observe what figures of these indices, when added together, will give a number an unit less than that expressing the number of terms; and find the product of the terms in the geometrical series which stand under these indices. This product multiplied by the first term given in the question, or the first term divided by this product, according as the progression is increasing or decreasing, will give the term sought.

Prob. II. Given one extreme, the ratio, and the number of terms, of a geometrical series, to find the sum of the terms.

Rule. Find the other extreme by Problem 1. Then divide the difference between the extremes by the ratio less 1; the quotient increased by the greater extreme will give the sum of the terms.

Prob. III. In any series of numbers in geometrical progression, decreasing ad infinitum, given the first term and the ratio, to find the sum of the series.

Rule. Subtract the second term from the first; the square of the first term, divided by this difference, will give the sum of the series.

If $l =$ the least term,
$g =$ the greatest,
$n =$ the number of terms.
$s =$ the sum of the terms.
$r =$ the ratio.
$log =$ logarithm of any letter.

Then will the following theorems exhibit all the possible cases of geometrical progression, including those already given.

86. If a series of quantities increase or decrease by continual multiplication or division by the same quantity, then those quantities are said to be in geometrical progression. For the exponents of r are a series of numbers in arithmetical progression, beginning with zero, or nought, and increasing by unity in each succeeding term, as 0, 1, 2, 3; and therefore the exponent of the nth term is $n-1$; consequently, if the nth term be represented by l, then will $l = ar^{n-1}$.

To find the sum of n terms of a geometrical progression, let $a =$ first term, r the common ratio, $n =$ the number of terms, and $s =$ the sum of (n) terms. Thus, the numbers 1, 2, 4, 8, 16, &c. which increase by continual multiplication by 2, and the numbers 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. which decrease by continual division by 3, or multiplication by $\frac{1}{3}$, are in geometrical progression, as before.

87. In general, if a represents the first term, r the common multiple or ratio, then may the series itself be represented by a, ar, a^2r, ar^3, ar^4, &c., which will evidently be an increasing or a decreasing series, according as r is a whole number or a proper fraction. In the foregoing series, the index of r in any term is less by unity than the number which denotes the place of that term in the series. Hence, if the number of terms in the series be denoted by n, the last term will be ar^{n-1}.

88. Let s be the sum of the series a, ar, ar^2, ar^3, &c.; then

$$a + ar + ar^2 + ar^3 + \text{\&c.} \;\ldots\ldots\; ar^{n-2} + ar^{n-1} = s.$$

Multiply the equation by r, and it becomes

$$ar + ar^2 + ar^3 + ar^4 + \text{\&c.}\ldots\ldots\; ar^{n-2} + ar^{n-1} + ar^n = rs.$$

Subtract the upper equation from the lower, or the series which is the value of s from the series which is the value of rs, and we have

$$ar^n - a = rs - s, \text{ or } (r-1)s = ar^n - a\,; \text{ and } s = \frac{ar^n - a}{r-1}.$$

If r is a proper fraction, then r and its powers are less than 1. For the convenience of calculation, therefore, it is better in this case to transform the equation, into $s = \frac{a - ar^n}{1-r}$, by multiplying the numerator and denominator of the fraction $\frac{ar^n - a}{r-1}$ by -1.

89. If l be the last term of a series of this kind, then $l = ar^{n-1}$; $\therefore rl = ar^n$; hence $s = \left\{\frac{ar^n - a}{r-1}\right\} = \frac{rl - r}{r-1} = l + \frac{l-a}{r-1}$.

From this equation, therefore, if any three of the four quantities s, a, r, l, be given, the fourth may be found. For $s = \frac{rl-a}{r-1}$; $a = rl - (r-1)s$; $r = \frac{s-a}{s-l}$, and $l = \frac{(r-1)s + a}{r}$. The value of n cannot be found from the equation $s = \frac{ar^n - a}{r-1}$, except by means of logarithms, thus: See page 96, Art. 92.

Find the sum of the series 1, 3, 9, 27, &c. to 12 terms.

Here $a=1$, $r=3$, $n=12$. and $s=$

$$\frac{ar^n - a}{r-1} = \frac{1 \times 3^{12} - 1}{3-1} = \frac{81^3 - 1}{2} = \frac{531441 - 1}{2} = \frac{531440}{2} = 265720.$$

90. Let l be the last term of an arithmetic series, whose first term is a, common difference d, and number of terms n; then

$l=a+(n-1)d$; $\therefore (n-1)d=l-a$, or $d=\frac{l-a}{n-1}$. Now the number of intermediate terms between the first and the last is $n-2$; let $n-2=m$, then $n-1=m+1$. Hence $d=\frac{l-a}{m+1}$, which gives the following rule for finding any number of arithmetic means between two numbers. Divide the difference of the two numbers by the given number of means increased by unity, and the quotient will be the common difference. Having the common difference, the means themselves will be known.

91. The general expression for the sum of a geometric series whose common ratio (r) is a fraction, is $s=\frac{a-ar^n}{1-r}$. Suppose now n to increase indefinitely; then r^n (r being a proper fraction) will decrease indefinitely; therefore ar^n will decrease indefinitely with respect to a, or a will be the limit of $a-ar^n$, and $\frac{a}{1-r}$ the limit of $\frac{a-ar^n}{1-r}$, or s; and consequently $\frac{a}{1-r}$ will express the value of the series when the number of its terms is supposed to be indefinitely increased, or, as it is commonly called, the sum of the series ad infinitum.

92. If the sum of the series, the common ratio, and the first term be given, the number of terms may be found thus:

Since $rs-s=ar^n-a$, or $ar^n=rs-s+a$, and $r^n=\frac{rs-s+a}{a}$.

$$\therefore \text{log. } r^n \text{ or } n\times\text{log. } r=\text{log. }(rs-s+a)-\text{log. } a.$$

$$\text{Hence } n=\frac{\text{log. }(rs-s+a)-\text{log. } a.}{\text{log. } r}.$$

1. The sum of a geometric series is 6560, its first term 2, and common ratio 3. What is the number of terms?

Here $s=6560$, $a=2$, $r=3$; $\therefore n=\frac{\text{log. }(rs-s+a)-\text{log. } a}{\text{log. } r}$

$$=\frac{\text{log. } 13122-\text{log.} 2}{\text{log. } 3.}=\frac{3.8169700}{.4771213}=8, \text{ Ans.}$$

2. A person undertakes a journey of 364 miles, going one mile the first day, three the second, nine the third, and so on. When will he arrive at his journey's end?

Here $s=364$, $a=1$, $r=3$; $\therefore n=\frac{\text{log. }(rs-s+a)-\text{log. } a}{\text{log. } r}$

$$\text{———}=\frac{\text{log. } 728-\text{log. } 3}{\text{log. } 1}=\frac{2.862728}{.477121}=6, \text{ days Ans.}$$

Demonstration of the Rule of Three.

Four numbers of the same kind are said to be proportional when the first contains the 2d as often as the 3d contains the 4th. If $\frac{A}{B}=\frac{C}{D}$; then $A:B::C:D$, or $B:A::D:C$. If 4 numbers are proportional, $A:B::C:D$, the product of the extreme is equal to the product of the means, viz. $A\times D=B\times C$. Let $\frac{A}{B}=\frac{C}{D}$; these fractions reduced, (Case 3, page 31,) become $\frac{AD}{BD}=\frac{BC}{BD}$; but when *equal* fractions have the same denominator their numerators are equal; therefore $AD=BC$. Hence we deduce a demonstration of the rule of three. If $A:B::C:D$; then $A\times D=B\times C$. Now if (p. 60, art. 14,) $\therefore D=\frac{B\times C}{A}$, which is the common rule as given in my Arithmetic.

Cor. $D=\frac{B}{A}\times C=\frac{C}{A}\times B=C\div\frac{A}{B}=B\div\frac{A}{C}$. The rule of 3 inverse may be made a rule of three direct, by making the 3d term the first, and by proceeding forward to the other two terms; $\therefore$ the above demonstration will serve for both rules.

Demonstration of the Double Rule of Three, or of Compound Proportion.

The rule of compound proportion is easily illustrated. Let C be any cause, and E the effect produced; and let cx be any other similar cause and ey the effect produced, then it is plain that $C:E::cx:ey$, $\therefore E\times cx=C\times ey$; hence x (an unknown part of the 2d cause) $=\frac{C\times ey}{E\times c}$, and $y=\frac{E\times cx}{C\times e}$(= an unknown part of the second effect. Hence the rule is manifest.

Harmonical Progression.

Three quantities are said to be in harmonical proportion when the first is to the third as the difference of the first and second is to the difference of the second and third.

Any number of magnitudes a, b, c, d, e, f, *&c.* are said to be in harmonical propartion, if $a:c::a-b:b-c$,
And $b:d::b-c:c-d$,
And $c:e::c-d:d-e$, *&c.*

The reciprocals of quantities in harmonical progression are in arithmetical progression. Let a, b, c, d, *&c.* be in harmonical progression; then since $a:c::a-b:b-c$, $\therefore ab-ac=ac-bc$. Divide this equation by abc: $\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}$. Again $b:d::b-c:$

$c-d$; $\therefore bc - bd = bd - dc$. Divide this equation by bcd, $\frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b}$, or $=\frac{1}{b}-\frac{1}{a}$; $\therefore$ the quantities $\frac{1}{a}$; $\frac{1}{b}$; $\frac{1}{c}$; $\frac{1}{d}$; &c. having a common difference, are in arithmetical progression.

From these definitions it follows, that in three harmonical proportionals, a, b, c, any two being given, the third may be found; since $ab-ac=ac-bc$, or $ab+bc=2ac$; $\therefore b=\frac{2ac}{a+c}$; that is, a harmonical mean between 2 quantities is equal to twice their product divided by their sum. Also, $c=\frac{ab}{2a-b}$ = a third harmonical proportion to a and b.

In a similar manner, if any three out of four harmonical proportions, a, b, c, d, be given, the other may be found; for since $a : d : a-b : c-d$, $\therefore ac-ad=ad-bd$; and from this equation I get $b=\frac{2ad-ac}{d}$; $c=\frac{2ad-bd}{a}$; $d=\frac{ac}{2a-b}$

The following is a curious property belonging to quantities in harmonical proportion, that their reciprocals are in arithmetical progression. Thus, if a, b, c, d, e, *&c.* be in harmonic proportion, then from what has been said above, I shall have $bc+ab=2ac$; $cd+bc=2bd$; $de+cd=2ce$, *&c.* Dividing the first of these equations by abc, the 2d by bcd, the 3d by cde, &c. I shall have $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$; $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$; $\frac{1}{c}+\frac{1}{e}=\frac{2}{d}$; $\therefore$ $\frac{1}{a}$; $\frac{1}{b}$; $\frac{1}{c}$; $\frac{1}{d}$; $\frac{1}{e}$; &c. are in arithmetical progression. Thus, of five musical strings of equal tension and thickness be made to sound together, the most perfect harmony they can produce, will be when their lengths are as $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$; $\frac{1}{5}$; $\frac{1}{6}$; or as 30, 20, 15, 12, 10; and hence this proportion is called harmonic, or musical proportion.

Problem 1. To insert any number (m) of harmonic means between two given quantities (a) and (b).

Since the reciprocals of quantities in harmonic progression are in arithmetical progression, therefore it will be best to insert (m) arithmetic means between $\frac{1}{a}$ and $\frac{1}{b}$; these means inverted will be the harmonic means required. Let x = the common difference of this arithmetic progression.

then $\frac{1}{a}+(m+1)x=\frac{1}{b}$; [p. 186.] & $x=\frac{a-b}{(m+1)ab}$; $\therefore$ the series $\frac{1}{a}$; $\frac{1}{a}+\frac{a-b}{(m+1)ab}$; $\frac{1}{a}+\frac{2(a-b)}{(m+1)ab}+\frac{1}{a}+\frac{3(a-b)}{(m+1)ab}$&c.....$\frac{1}{b}$ is arith.;

and the reciprocals a; $\frac{(m+1)ab}{a+mb}$; $\frac{(m+1)ab}{2a+(m-1)b}$; $\frac{(m+1)ab}{3a+(m-2)b}$; are in harmonic progression. Insert 6 harmonic means between 1 and 20. Here $m=6$, $a=1$, $b=20$; ∴ the means are $\frac{140}{121}$; $\frac{70}{51}$; $\frac{140}{83}$; $\frac{35}{16}$; $\frac{28}{8}$; $\frac{70}{13}$.

Prob. 2. Having given two consecutive terms of an harmonical series to determine the law and limit of continuation.

Let t, t', be two consecutive terms of the harmonic progression, and let a, $a-x$, be the two consecutive terms of the corresponding arithmetic progression; then

$\frac{1}{t}=a$; $\frac{1}{t'}=a-x$; therefore $x=\frac{t'-t}{tt'}$, and since the (n)th or general term of the arithmetic progression, reckoning from $\frac{1}{t}$ is $(a\mp nx)$, therefore the (n) term of the harmonic progression is

$\frac{1}{a\mp nx}=\frac{tt'}{t'\mp n(t'-t)}$, which is indefinite when $t'-n(t'-t)=0$, or when $n=\frac{t'}{t'-t}$.

Given 3, 4, two consecutive terms of an harmonic progression, to show the law of continuation both ways. The general law is $\frac{12}{4\mp n}$; ∴ the next term of the progression after 4, is $\frac{12}{4-2}$, or 6; the next is $\frac{12}{4-3}$ or 12; the next $\frac{12}{4-4}$ or $\frac{12}{0}$, a quantity indefinitely great. The term preceding 3 is $\frac{12}{4+1}$ or $\frac{12}{5}$; the next preceding term $\frac{12}{6}$ or 2, &c. hence the series cannot be carried beyond 12, but may be extended to any length the other way.

An arithmetic mean between two quantities is greater than a geometric, and a geometric is greater than a harmonic mean. Let p and q be the two quantities, then since $(p-q)^2$ is positive, whether p be greater or less than q, ∴ $p^2-2pq+q^2$ is positive, or p^2+q^2 greater than $2pq$, ∴ $p^2+2pq+q^2$ greater than $4pq$ or $p+q$ greater than $2\sqrt{(pq)}$; ∴ $\frac{1}{2}(p+q)$ is greater than $\sqrt{(pq)}$. But $\frac{1}{2}(p+q)$ is an arith. mean between a and b, and $\sqrt{(pq)}$ is a geo. mean, ∴ the arith. is greater than the geo. mean. Again $\frac{2pq}{p+q}$ is an harmonic mean between p and q. Also $\frac{p+q}{2}:\sqrt{(pq)}::\sqrt{(pq)}:\frac{2pq}{p+q}$

Or arith . mean : geo. mean : :geo. mean : harmonic mean. But since the first term has been proved to be greater than the second, ∴ by Euclid, book 5, p. A., the 3d term is greater than the fourth.

1. A man buys 80 pounds of pepper and 36 pounds of saffron so that for 8 dollars he had 14 pounds of pepper more than he had of saffron for 26 dollars, and what he laid out amounted to 188 dollars. How many pounds of pepper had he for 8 dollars, and how many of saffron for 26 dollars? Ans. P 20, and 6 S.

Surds.

The meaning of surd quantities, and the method of representing them, have already been explained in pages 50, 51, and 52. *Simple Surds* are such as consist only of one term, as $\sqrt{2}$, $\sqrt[3]{7}$, and $\sqrt[4]{5}$, &c. *Compound Surds* are such as consist of two or more terms, as $\sqrt{3}+\sqrt{3}$; $\sqrt{5}+\sqrt{3}-\sqrt{2}$, &c.; and when a quantity of this kind consists only of two terms, it is called a binomial surd.

Case I. *To reduce a rational quantity to the form of a surd.*

Rule. Raise the quantity to the power denoted by the root of the surd, and over this new quantity place the radical sign, or proper index, and it will be the form required.

1. Reduce 5 to the form of the square root.

First $5\times 5=25$; whence the form is $\sqrt{25}$, the answer.

2. Reduce 3 to the form of the square root.

First $3\times 3=3^2=9$, consequently $\sqrt{9}$ is the form.

3. Reduce $\frac{2}{3}$ and $a-b$ to the form of the cube root.

Here $\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{8}{27}=\sqrt[3]{\frac{8}{27}}$, and $(a-b)(a-b)(a-b)=(a^3-b^3)=\sqrt[3]{(a^3-b^3)}$, and therefore the answers are $\sqrt[3]{\frac{8}{27}}$, and $\sqrt[3]{(a^3-b^3)}=\sqrt[3]{(a-b)^3}$.

Case II. *The reduction of quantities of different indices, to othere that shall have a given index, or to reduce surds of different indices to others of equal value having the same index.*

Rule 1. Divide the indices of the proposed by the given index, and the quotient will be the new indices for those quantities; or reduce the feactional indices to a common denominator, which will be the index required; then raise each of the quantities to the power denoted by its numerator.

1. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to the same index.

Here the fractional indices are $\frac{1}{2}$ and $\frac{1}{3}$, and these reduced to a common denominator become $\frac{3}{6}$ and $\frac{2}{6}$, therefore $3^{\frac{1}{2}}=3^{\frac{3}{6}}=(3^3)^6=27^{\frac{1}{6}}$, or $\sqrt[3]{27}$; and $2^{\frac{1}{3}}=2^{\frac{2}{6}}=(2^2)^6=4^{\frac{1}{6}}$, or $\sqrt[6]{4}$, which are the surds with the same index $\sqrt{\ }$.

2. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to quantities that shall have a common index.

Here $\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$; therefore $(a^3)^{\frac{1}{6}}$ and $(a^2)^{\frac{1}{6}}$, Ans.

Here again $\frac{1}{3}=\frac{1}{12}$ and $\frac{1}{4}=\frac{3}{12}$; therefore $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$.

2. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to quantities that shall have a common index.

Here $\frac{1}{n}$, $\frac{1}{m}$ reduces to the common denominators $\frac{m}{mn}$ and $\frac{n}{mn}$; therefore $(a^m)^{\frac{1}{mn}}$ and $(b^n)^{\frac{1}{mn}}$, Ans.

Case III. *To reduce surds to their most simple forms.*

Rule. Resolve the given number, or quantity, into two factors, one of which shall be the greatest power contained in it, and set the root of this power before the remaining part, with the proper radical sign between them.*

1. Let $\sqrt{48}$ and $\sqrt[3]{108}$ each be reduced to its most simple form. Here $\sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}$ and $\sqrt[3]{108}=\sqrt[3]{(27\times4)}=3\sqrt[3]{4}$.

Note 1. When any number, or quantity, is prefixed to the surd, that quantity must be multiplied by the root of the factor above-mentioned, and the product then joined to the other part, as before.

1. Let $2\sqrt{32}$ and $5\sqrt[3]{24}$ each be reduced to its most simple form. Here $2\sqrt{32}=2\sqrt{(16\times2)}=8\sqrt{2}$, & $5\sqrt[3]{24}=5\sqrt[3]{(8\times3)}=10\sqrt[3]{3}$.

Note 2. A fractional surd may also be reduced to a more convenient form, by multiplying both the numerator and denominator by such a number, or quantity, as will make the denominator a complete power of the kind required; and then joining its root, with 1 put over it, as a numerator, to the other part of the surd.

1. Let $\sqrt{\frac{2}{7}}$, $\sqrt[3]{\frac{2}{5}}$, $\sqrt{125}$, $\sqrt{294}$, $\sqrt[3]{56}$, $\sqrt[3]{192}$, $7\sqrt{80}$, $9\sqrt[3]{81}$, and $\frac{3}{121}\sqrt{\frac{5}{6}}$, each be reduced to its most simple form.

Here $\sqrt{\frac{2}{7}}=\sqrt{\frac{14}{49}}=\sqrt{(\frac{1}{49}\times14)}=\frac{1}{7}\sqrt{14}$, Ans. $3\sqrt[3]{\frac{2}{5}}=3\sqrt[3]{\frac{50}{125}}=3\sqrt[3]{(\frac{1}{125}\times50)}=\frac{3}{5}\sqrt[3]{50}$, Ans. $\sqrt{125}=\sqrt{(25\times5)}=5\sqrt{5}$, Ans. $\sqrt{294}=\sqrt{(49\times6)}=7\sqrt{5}$, Ans. $\sqrt[3]{56}=\sqrt[3]{(8\times7)}=2\sqrt[3]{7}$, Ans. $\sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}$, A. $7\sqrt{80}=7\sqrt{(16\times5)}=28\sqrt{5}$, A. $9\sqrt[3]{81}=9\sqrt[3]{(27\times3)}=27\sqrt[3]{3}$, Ans.

Here, reducing the radical, we have $\sqrt{\frac{5}{6}}=\sqrt{\frac{5\times6}{36}}=\frac{1}{6}\sqrt{30}$; therefore $\frac{3}{121}\sqrt{\frac{5}{6}}=\frac{3}{121}\times\frac{1}{6}\sqrt{30}=\frac{1}{242}\sqrt{30}$, Ans.

10. Reduce $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$, $\sqrt{98a^2x}$, and $\sqrt{(x^3-a^2x^2)}$, each to the most simple form.

Here $\sqrt[3]{\frac{3}{16}}=\sqrt[3]{\frac{3\times4}{64}}=\frac{1}{4}\sqrt[3]{12}$; hence $\frac{4}{7}\sqrt[3]{\frac{3}{16}}=\frac{1}{7}\sqrt[3]{12}$, Ans.

$\sqrt{98a^2x}=(49a^2\times2x)=7a\sqrt{2x}$, Ans.

$\sqrt{(x^3-a^2x^2)}=\sqrt{\{x^2(x-a^2)\}}=x\sqrt{(x-a^2)}$, Ans.

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* When the given surd contains no factor that is an exact power of the kind required, it is already in its most simple form. Thus, $\sqrt{15}$ cannot be reduced lower, because neither of its factors, 5 nor 3, is a square.

Case IV. *To add Surd Quantities together.*

Rule. When the surds are of the same kind, reduce them to their simplest forms, as in the last case; then, if the surd part be the same in each of them, annex it to the sum of the rational parts, and it will give the whole sum required. But if the quantities have different indices, or the surd part in each be not the same, they can only be added together by the signs + and —.

1. It is required to find the sum of $\sqrt{27}$ and $\sqrt{48}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}\\ \text{And } \sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}\end{array}\right\}=7\sqrt{3}\text{ the sum.}$$

2. It is required to find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

$$\left.\begin{array}{l}\text{Here } \sqrt[3]{500}=\sqrt[3]{(125\times4)}=5\sqrt[3]{4}\\ \text{And } \sqrt[3]{108}=\sqrt[3]{(27\times4)}=3\sqrt[3]{4}\end{array}\right\}=8\sqrt[3]{4}\text{ the sum.}$$

3. Find the sum of $\sqrt{50}$ and $\sqrt{72}$, or $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{1}{15}}$. or

$$\left.\begin{array}{l}\sqrt{50}=\sqrt{(25\times2)}=5\sqrt{2}\\ \sqrt{72}=\sqrt{(36\times2)}=6\sqrt{2}\end{array}\right\} \qquad \left\{\begin{array}{l}\sqrt{\frac{3}{5}}=\sqrt{\frac{15}{25}}=\frac{1}{5}\sqrt{15}\\ \sqrt{\frac{1}{15}}=\sqrt{\frac{15}{225}}=\frac{1}{15}\sqrt{15}\end{array}\right.$$

$$\text{Ans. } 11\sqrt{2}. \qquad \text{Ans. } \tfrac{4}{15}\sqrt{15}.$$

Case V. *To find the difference of Surd Quantities.*

Rule. When the surds are of the same kind, prepare the quantities as in the last rule; then the difference of the rational parts annexed to the common surd will give the whole difference required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be subtracted by means of the sign —.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{448}=\sqrt{(64\times7)}=8\sqrt{7}\\ \text{And } \sqrt{112}=\sqrt{(16\times7)}=4\sqrt{7}\end{array}\right\}=4\sqrt{7}\text{ the difference.}$$

2. Required to find the difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

$$\left.\begin{array}{l}\text{Here } \sqrt[3]{192}=\sqrt[3]{(64\times3)}=4\sqrt[3]{3}\\ \text{And } \sqrt[3]{24}=\sqrt[3]{(8\times3)}=2\sqrt[3]{3}\end{array}\right\}=2\sqrt[3]{3}\text{, the diff.}$$

3. Find the difference of $\frac{3}{4}\sqrt{\frac{2}{3}}$, and $\frac{2}{5}\sqrt{\frac{1}{6}}$.

$$\left.\begin{array}{l}\text{Here } \frac{3}{4}\sqrt{\frac{2}{3}}=\frac{3}{4}\sqrt{\frac{6}{9}}=\frac{3}{12}\sqrt{6}=\frac{1}{4}\sqrt{6}\\ \text{And } \frac{2}{5}\sqrt{\frac{1}{6}}=\frac{2}{5}\sqrt{\frac{6}{36}}=\frac{2}{30}\sqrt{6}=\frac{1}{15}\sqrt{6}\end{array}\right\}=\tfrac{11}{60}\sqrt{6}\text{, the diff.}$$

4. Find the difference of $2\sqrt{\frac{1}{2}}$ and $\sqrt{8}$, or $\sqrt[3]{3\frac{1}{3}}$ and $\sqrt[3]{72}$.

$$\left.\begin{array}{l}\text{Here } 3\sqrt[3]{\frac{1}{3}}=3\sqrt[3]{\frac{9}{27}}=\sqrt[3]{9}\\ \text{And } \sqrt[3]{72}=\sqrt{(8\times9)}=2\sqrt[3]{9}\end{array}\right\} \qquad \left\{\begin{array}{l}\text{Here } 2\sqrt{\frac{1}{2}}=\sqrt{\frac{2}{4}}=2\sqrt{2}\\ \text{And } \sqrt{8}=(4\times2)=2\sqrt{2}\end{array}\right.$$

$$\text{the diff. }=\sqrt[3]{9} \qquad \text{the diff. }=\sqrt{2}$$

5. Find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.

$$\left.\begin{array}{l}\text{Here } \sqrt{80a^4x}=\sqrt{(16a^4\times5x)}=4a^2\sqrt{5x}\\ \text{And } \sqrt{20a^2x^3}=\sqrt{(4a^2x^2\times5x)}=2ax\sqrt{5x}\end{array}\right\}=(4a^2-2ax)\sqrt{5x}.$$

6. Find the difference of $8\sqrt[3]{a^3b}$ and $2\sqrt[3]{a^6b}$.

$$\left.\begin{array}{l}\text{Here } 8\sqrt[3]{a^3b}=8\sqrt[3]{(a^3\times b)}=8a\sqrt[3]{b}\\ \text{And } 2\sqrt[3]{a^6b}=2\sqrt[3]{(a^6\times b)}=2a^2\sqrt[3]{b}\end{array}\right\}=(8a-2a^2)\sqrt[3]{b}\text{, diff.}$$

7. Find the difference of $\sqrt{75}$ and $\sqrt{48}$, or $\sqrt[3]{256}$ and $\sqrt[3]{32}$.

$$\left.\begin{array}{l}\sqrt{75}=\sqrt{(25\times3)}=5\sqrt{3}\\ \sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}\end{array}\right\}\quad\left\{\begin{array}{l}\sqrt[3]{256}=\sqrt[3]{(64\times4)}=4\sqrt[3]{4}\\ \sqrt[3]{32}=\sqrt[3]{(\ 8\times4)}=2\sqrt[3]{4}\end{array}\right.$$

$$\text{difference, }\sqrt{3}\qquad\qquad \text{difference, }2\sqrt[3]{4}$$

8. Find the difference of $\sqrt{\frac{3}{4}}$ and $\sqrt{\frac{1}{3}}$, or $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{3}{8}}$.

$\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}$, and $\sqrt{\frac{1}{3}}=\sqrt{(\frac{1}{3}\times\frac{3}{3})}=\frac{1}{3}\sqrt{3}$; then $(\frac{1}{2}-\frac{1}{3})\sqrt{3}=\frac{1}{6}\sqrt{3}$, $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{1}{3}\sqrt{6}$, and $\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\frac{1}{4}\sqrt{6}$; $(\frac{1}{3}-\frac{1}{4})\sqrt{6}=\frac{1}{12}\sqrt{6}$,

Case VI. *To multiply Surd Quantities together.*

Rule. When the surds are of the same kind, find the product of the rational parts, and the product of the surds, and the two joined together, with the common radical sign between them, will give the whole product required; which may be reduced to its most simple form by Case III.

But if the surds are of different kinds, they must be reduced to a common index, and then multiplied together as usual.

It is also to be observed, as before mentioned, that the product of different powers, or roots, of the same quantity, is found by adding their indices.

1. Required the product of $\frac{1}{2}\sqrt{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

$\frac{1}{2}\sqrt[3]{\frac{2}{3}}\times\frac{3}{4}\sqrt[3]{\frac{5}{6}}=\frac{3}{8}\sqrt[3]{\frac{5}{9}}=\frac{3}{8}\sqrt[3]{\frac{25}{27}}=\frac{1}{8}\sqrt[3]{15}$, Ans.

2. Find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

$3\sqrt{8}\times2\sqrt{6}=6\sqrt{48}=6\sqrt{(16\times3)}=24\sqrt{3}$, Ans.

3. Find the products of $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$, or $5\sqrt{a}$ and $3\sqrt[3]{a}$.

$$\left.\begin{array}{l}\text{Here } 2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}\\ \text{And } 3^{\frac{1}{3}}=3^{\frac{2}{6}}=(3^2)^{\frac{1}{6}}=9^{\frac{1}{6}}\end{array}\right\}\quad\left\{\begin{array}{l}\text{Here } 5\sqrt{a}=5a^{\frac{1}{2}}=5a^{\frac{3}{6}}\\ \text{And } 3\sqrt[3]{a}=3a^{\frac{1}{3}}=3a^{\frac{2}{6}}\end{array}\right.$$

Ans. $(8^{\frac{1}{6}}\times9^{\frac{1}{6}})=(72)^{\frac{1}{6}}$, Ans. $15a^{\frac{5}{6}}=15(a^5)^{\frac{1}{6}}$, or $15\sqrt[6]{a^5}$

4. Required to find the product of $5\sqrt{8}$ and $3\sqrt{5}$.

$5\sqrt{8}\times3\sqrt{5}=15\sqrt{40}=15\sqrt{(4\times10)}=30\sqrt{10}$, Ans.

5. Required the product of $4+2\sqrt{2}$ and $2-\sqrt{2}$.

6. Required the product of $(a+b)^{\frac{1}{n}}$ and $(a+b)^{\frac{1}{m}}$.

$$\begin{array}{ll}\text{5. Mult.} & 4+2\sqrt{2}\\ \text{By} & \underline{2-\sqrt{2}}\\ & 8+4\sqrt{2}\\ & \underline{\quad-4\sqrt{2}-4}\\ \text{Ans.} & 8-4=4\end{array}\qquad\begin{array}{ll}\text{6. Mult.} & (a+b)^{\frac{1}{n}}=(a+b)^{\frac{m}{mn}}\\ \text{By} & \underline{(a+b)^{\frac{1}{m}}=(a+b)^{\frac{n}{mn}}}\\ \text{Product} & =(a+b)^{\frac{m+n}{mn}}.\end{array}$$

Multiply $2\sqrt[3]{14}$ by $3\sqrt[3]{4}$, or $2a^{\frac{2}{3}}$ by $a^{\frac{4}{3}}$, or $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$.

$2\times3\sqrt[3]{14}\times\sqrt[3]{4}=6\sqrt[3]{56}=6\sqrt[3]{(8\times7)}=12\sqrt[3]{7}$, Ans.

$(a+b)^{\frac{1}{3}}\times(a+b)^{\frac{3}{4}}=(a+b)^{\frac{4}{12}}\times(a+b)^{\frac{9}{12}}=(a+b)^{\frac{4+9}{12}=\frac{13}{12}}$, Ans.

Multiply $2x+\sqrt{b}$ and $2x-\sqrt{b}$, or $2x^{\frac{1}{4}}$ and $3x^{\frac{1}{4}}$.

$$\text{Or } 2\times1\times a^{\frac{2}{3}}\times a^{\frac{4}{3}}=2a^{\frac{6}{3}}=2a^{2}, \text{ Ans.}$$

Multiply $(a+2\sqrt{b})^{\frac{1}{2}}$ by $(a-2\sqrt{b})^{\frac{1}{2}}$, or $4x^{\frac{1}{n}}$ by $2y^{\frac{1}{n}}$.

$$\begin{array}{l} 2x+\sqrt{b} \\ 2x-\sqrt{b} \\ \hline 4x^2+2x\sqrt{b} \\ \quad -2\sqrt{xb}-b \\ \hline 4x^2- \qquad b, \text{ Ans.} \end{array} \qquad \begin{array}{l} (a+2\sqrt{b})^{\frac{1}{2}} \\ (a-2\sqrt{b})^{\frac{1}{2}} \\ \hline a^2+2a\sqrt{b} \\ \quad -2a\sqrt{b}-4b \\ \hline (a^2 \qquad -4b)^{\frac{1}{2}}, \text{ Ans.} \end{array} \qquad \begin{array}{l} 2x^{\frac{1}{n}} \\ 3x^{\frac{1}{n}} \\ \hline 6(x^2)^{\frac{1}{n}}=6\sqrt[n]{x^2}, \text{Ans.} \end{array}$$

$$\text{Here } 4x^{\frac{1}{n}}\times2y^{\frac{1}{n}}=8(xy)^{\frac{1}{n}}, \text{ or }=8\sqrt[n]{xy}, \text{ Ans.}$$

CASE VII. *To divide one Surd Quantity by another.*

RULE. When the surds are of the same kind, find the quotient of the rational parts, and the quotient of the surds, and the two joined together, with the common radical sign between them, will give the whole quotient required.

But if the surds are of different kinds, they must be reduced to a common index, and then be divided as before.

It is also to be observed, that the quotients of different powers or roots of the same quantity, are found by subtracting their indices.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\dfrac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}=4\sqrt{(9\times2)}=12\sqrt{2}$, Ans.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt{2}$.

Here $\dfrac{8\sqrt[3]{512}}{4\sqrt[3]{2}}=2\sqrt[3]{256}=2\sqrt[3]{(64\times4)}=8\sqrt[3]{4}$, Ans.

3. It is required to divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$, and $\sqrt{7}$ by $\sqrt[3]{7}$.

$$\text{Here } \frac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}}=\tfrac{3}{2}\sqrt{\tfrac{5}{2}}=\tfrac{3}{2}\sqrt{\tfrac{10}{4}}=\tfrac{3}{4}\sqrt{10}, \text{ Ans.}$$

$$\text{And } \frac{\sqrt{7}}{\sqrt[3]{7}}=\frac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}}=\frac{7^{\frac{3}{6}}}{7^{\frac{2}{6}}}=7^{\frac{3}{6}-\frac{2}{6}}=7^{\frac{1}{6}}, \text{ Ans.}$$

4. Required to divide $6\sqrt{54}$ by $3\sqrt{2}$, and $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.

Here $\dfrac{6\sqrt{54}}{3\sqrt{2}}=2\sqrt{27}=2\sqrt{(9\times3)}=6\sqrt{3}$, Ans.

And $\dfrac{4\sqrt[3]{72}}{2\sqrt[3]{18}}=2\sqrt[3]{4}$, which will not reduce lower.

5. Required to divide $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$.

Here $5\frac{3}{4}\div\frac{2}{3}=\frac{23}{4}\times\frac{3}{2}=\frac{69}{8}$, and $\sqrt{\frac{1}{135}}\div\sqrt{\frac{1}{5}}=\sqrt{\frac{1}{27}}=\sqrt{\frac{3}{81}}$ $=\frac{1}{9}\sqrt{3}$; $\therefore\ \frac{69}{8}\times\frac{1}{9}\sqrt{3}=\frac{69}{72}\sqrt{3}=\frac{23}{24}\sqrt{3}$, Ans.

6. Divide $4\frac{5}{7}\sqrt{\frac{2}{3}}$ by $2\frac{2}{5}\sqrt{\frac{3}{4}}$, and $4\frac{1}{2}\sqrt[2]{a}$ by $2\frac{2}{3}\sqrt[3]{ab}$.

First $4\frac{5}{7}\div2\frac{2}{5}=\frac{33}{7}\times\frac{5}{12}=\frac{165}{84}$, and $\sqrt{\frac{2}{3}}\div\sqrt{\frac{3}{4}}=\sqrt{(\frac{2}{3}\times\frac{4}{3})}=\sqrt{\frac{8}{9}}$ $=\sqrt{\frac{4\times2}{9}}=\frac{2}{3}\sqrt{2}$; hence $\frac{165}{84}\times\frac{2}{3}\sqrt{2}=\frac{55}{42}\sqrt{2}$. Ans.

Here $4\frac{1}{2}\div2\frac{2}{3}=\frac{9}{2}\times\frac{3}{8}=\frac{27}{16}$, and $\sqrt{a}\div\sqrt[3]{ab}=a^{\frac{1}{2}}\div a^{\frac{1}{3}}b^{\frac{1}{3}}=$ $\frac{a^{\frac{1}{6}}}{b^{\frac{1}{3}}}=\frac{a^{\frac{1}{6}}}{b^{\frac{2}{6}}}=(\frac{a}{b^2})^{\frac{1}{6}}$; therefore $\frac{27}{16}(\frac{a}{b^2})^{\frac{1}{6}}$, the answer.

7. Required to divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{a}$.

Here $32\frac{2}{5}\div13\frac{3}{4}=\frac{162}{5}\times\frac{4}{55}=\frac{648}{275}$, and $\sqrt{a}\div\sqrt[3]{a}=a^{\frac{1}{2}}\div a^{\frac{1}{3}}=$ $a^{\frac{3}{6}-\frac{2}{6}}=a^{\frac{1}{6}}$. Therefore $\frac{648}{275}a^{\frac{1}{6}}$, the answer.

8. Required to divide $9\frac{3}{8}a^{\frac{1}{n}}$ by $4\frac{9}{11}a^{\frac{1}{m}}$.

Here $9\frac{3}{8}\div4\frac{9}{11}=\frac{75}{8}\times\frac{11}{53}=\frac{825}{424}$,

And $a^{\frac{1}{n}}\div a^{\frac{1}{m}}=a^{\frac{m}{mn}}\div a^{\frac{n}{mn}}=a^{\frac{m-n}{mn}}$; consequently, $\frac{825}{424}a^{\frac{m-n}{mn}}$, Ans.

9. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$.

Here $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}+\sqrt{3}}=\sqrt{4}$, or 2, Ans.

NOTE 1. Since the division of surds is performed by subtracting their indices, it is evident that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator, by changing the sign of its index, or exponent.

2. It likewise appears, from the same rule, that

$$\frac{a^m}{a^m}=a^{m-m}=a^0\text{; but we have also }\frac{a^m}{a^m}=1.$$

Whence it follows, that a^0 is a symbol equivalent to unity, and may, consequently, be always replaced by 1, whenever it occurs.*

CASE VIII. *To involve or raise Surd Quantities to any power.*

RULE. When the surd is a simple quantity, multiply its index by 2 for the square, 3 for the cube, &c. and it will give the power of the surd part; which, being annexed to the proper power of the rational parts, will give the whole power required.

And if it be a compound quantity, multiply it by itself, the proper number of times, according to the usual rule.†

1. Thus $\frac{1}{a}=\frac{a^{-1}}{1}$ or a^{-1}; and $\frac{1}{a^n}=\frac{a^{-n}}{1}$, or a^{-n}.

2. Also $\frac{b}{a^2}=\frac{ba^{-2}}{1}$, or ba^{-2}; and $\frac{a^{-n}}{b^{-m}}=\frac{1}{a^nb^{-m}}$, or $\frac{b^m}{a^n}$.

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* This expression, properly speaking, is only the result of an

3. Let $\frac{1}{a^2}$ be expressed with a negative index.

4. Let $a^{-\frac{1}{2}}$ be expressed with a negative index.

Here $\frac{1}{a^2}=a^{-2}=a^{-2}$ the answer. Here $a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}$, Ans.

5. Let $\frac{1}{a+x}$ be expressed with a negative index.

Here $\frac{1}{a+x}=\frac{1}{(a+x)^1}=(a+x)^{-1}$, Ans.

6. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed with a negative index.

Here $a(a^2-x^2)^{-\frac{1}{3}}=\frac{a}{(a^2-x^2)^{\frac{1}{3}}}$, Ans. or $\frac{1}{a(a^2-y^2)^{\frac{1}{3}}}$

7. Required to find the square of $\frac{2}{3}a^{\frac{1}{3}}$, and the cube of $\frac{2}{3}\sqrt{3}$.

Here $\left\{\frac{2}{3}a^{\frac{1}{3}}\right\}2=\frac{4}{9}a^{\frac{1}{3}\times 2}=\frac{4}{9}a^{\frac{2}{3}}=\frac{4}{9}\sqrt[3]{a^2}$, Ans.

And $\frac{8}{27}=3^{\frac{3}{2}}=\frac{8}{27}\sqrt{27}=\frac{8}{27}\sqrt{9\times 3}=\frac{8}{9}\sqrt{3}$, Ans.

8. Required to find the square of $3\sqrt[3]{3}$, and the cube of $17\sqrt{21}$.

Here $(3\sqrt{2})^2=(3\times 3^{\frac{1}{3}})^2=9.3^{\frac{2}{3}}=9\sqrt[3]{9}$, Ans.

$(17\sqrt{21})^3=17^3\sqrt{(21^2+21)}=17^3\times 21\sqrt{21}=103173\sqrt{21}$, Ans.

9. Required to find the 4th power of $\frac{1}{6}\sqrt{6}$. Ans. $\frac{1}{36}$.

Here $(\frac{1}{6}\sqrt{6})^4=(\frac{1}{6}\times 6^{\frac{1}{2}})^2=\frac{1}{6^4}\times 6^2=\frac{1}{6^2}$; therefore $\frac{1}{36}$, Ans.

10. Find the square of $3+2\sqrt{5}$, the cube of $\sqrt{x}+3\sqrt{y}$, and the 4th power of $\sqrt{3}-\sqrt{2}$.

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operation, to which we are led by the common practice of representing the powers of quantities by numeral indices; which, in the present case, gives the quotient a^0, instead of 1, as it would be according to the usual method of division.

† Any power, or root, of a fraction, as before observed, is equal to the same power, or root, of the numerator, divided by the like power, or root, of the denominator. Thus,

$$(\frac{a}{b})^2=\frac{a^2}{b^2}, \text{ and } (\frac{a}{b})^m=\frac{a^m}{b^m}; \text{ also } \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{ab}}{b}, \text{ and } (\frac{a}{b})^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

And if any quantity that is affected with the sign of the square root, is to be raised to the second power, it is done by suppressing the sign. Thus, $(\sqrt{a})^2$, or $\sqrt{a}\times\sqrt{a}=a$, and $\{\sqrt{(a+b)}\}^2$, or $\sqrt{(a+b)}\times\sqrt{(a+b)}=a+b$.

$$\begin{array}{r}
\sqrt{3}-\sqrt{2} \\
\sqrt{3}-\sqrt{2} \\ \hline
3-\sqrt{6} \quad\quad \\
-\sqrt{6}+2 \\ \hline
5-\ 2\sqrt{6} \quad\quad \\
5-\ 2\sqrt{6} \quad\quad \\ \hline
25-\ 10\sqrt{6} \quad\quad \\
-\ 10\sqrt{6}+24 \\ \hline
49-\ 20\sqrt{6},\ \text{Ans.}
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & \sqrt{x}+3\sqrt{y} \\
\text{By} & \sqrt{x}+3\sqrt{y} \\ \hline
& x+3\sqrt{xy} \\
& \quad +3\sqrt{xy}+9y \\ \hline
& x+6\sqrt{xy}+9y \\
& \sqrt{x}+3\sqrt{y} \\ \hline
& x\sqrt{x}+6x\sqrt{y}+\ 9y\sqrt{x} \\
& \quad +3y\sqrt{y}+18y\sqrt{x}+27y\sqrt{y} \\ \hline
& x\sqrt{x}+9x\sqrt{y}+27y\sqrt{x}+27y\sqrt{y}=\text{Cube.}
\end{array}$$

$$\begin{array}{r}
3+2\sqrt{5} \\
3+2\sqrt{5} \\ \hline
9+6\sqrt{5} \quad\quad\quad \\
+9\sqrt{5}+4\sqrt{25} \\ \hline
9+12\sqrt{5}+20 \\
\text{or } 29+12\sqrt{5}
\end{array}$$

CASE IX. *To find the roots of Surd Quantities.*

RULE. When the surd is a simple quantity, multiply its index by $\frac{1}{2}$ for the square root, by $\frac{1}{3}$ for the cube root, &c. and it will give the root of the surd part; which being annexed to the root of the rational part, will give the whole root required. And if it be a compound quantity, find its root by the usual rule.

1. Find the square root of $9\sqrt[3]{3}$, and the cube root of $\frac{1}{8}\sqrt{2}$.

Here $(9\sqrt{3})^{\frac{1}{2}}=9^{\frac{1}{2}}\times2^{\frac{1}{3}\times\frac{1}{2}}=9^{\frac{1}{2}}\times3^{\frac{1}{6}}=\sqrt[6]{3}$, Ans.

And $(\frac{1}{8}\sqrt{2})^{\frac{1}{3}}=(\frac{1}{8})^{\frac{1}{3}}\times(2^{\frac{1}{2}\times\frac{1}{3}})=\frac{1}{2}(2^{\frac{1}{6}})=\frac{1}{2}\sqrt[6]{2}$, Ans.

2. It is required to find the square root of 10^3.

Here $\sqrt{10^3}=\sqrt{(10^2+10)}=10\sqrt{10}$, Ans. $10\sqrt{(10)}$.

3. Find the cube root of $\frac{8}{27}a^4$, and the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

$\sqrt{\frac{8}{27}a^4}=\sqrt{(\frac{8}{27}a^3\times a)}=\frac{2}{3}a\sqrt[3]{a}$, and $\frac{a}{3}\sqrt{\frac{a}{3}}=(\frac{a}{3})^1\times(\frac{a}{3})^{\frac{1}{2}}=(\frac{a}{3})^{\frac{3}{2}}$

Consequently $\sqrt[3]{(\frac{a}{3}\sqrt{\frac{a}{3}})}=\sqrt[3]{(\frac{a}{3})^{\frac{3}{2}}}=(\frac{a}{3})^{\frac{3}{6}}=(\frac{3a}{9})^{\frac{1}{2}}=\frac{1}{3}\sqrt{3a}$.

4. It is required to find the 4th root of $\frac{16}{81}a^{\frac{2}{3}}$.

Here $\sqrt{\frac{16}{81}a^{\frac{2}{3}}}=\frac{2}{3}a^{\frac{2}{3}\times\frac{1}{4}}=\frac{2}{3}a^{\frac{1}{6}}$, Ans.

5. Find the square roots of $x^2-4x\sqrt{a}+4a$, and $a+2\sqrt{ab}+b$.

$$\begin{array}{rl}
\text{Here} & a+2\sqrt{ab}+b(\sqrt{a}+\sqrt{b} \\
& a \\ \hline
2\sqrt{a}+\sqrt{b}) & 2\sqrt{ab}+b \\
& 2\sqrt{ab}+b
\end{array}
\qquad
\begin{array}{rl}
& x^2-4x\sqrt{a}+4a(x-2\sqrt{a} \\
& x^2 \quad\quad\quad\quad\quad\quad \text{Ans.} \\ \hline
2x-2\sqrt{\ }) & -4x\sqrt{a}+4a \\
& -4x\sqrt{a}+4a
\end{array}$$

CASE X. *To transform a Binomial, or a Residual Surd, into a General Surd.*

RULE. Involve the given binomial, or residual, to a power corresponding with that denoted by the surd; then set the radical sign of the same root over it, and it will be the general surd required.

1. It is required to reduce $2+\sqrt{3}$ to a general surd.
Here $(2+\sqrt{3})^2=4+3+4\sqrt{3}=7+4\sqrt{3}$; therefore $2+\sqrt{3}=\sqrt{(7+4\sqrt{3})}$, the answer.

2. It is required to reduce $\sqrt{2}+\sqrt{3}$ to a general surd.
Here $(\sqrt{2}+\sqrt{3})^2=2+3+2\sqrt{6}=5+2\sqrt{6}$; therefore
$\sqrt{2}+\sqrt{3}=\sqrt{(5+2\sqrt{6})}$, the answer.

3. It is required to reduce $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.
Here $(\sqrt[3]{2}+\sqrt[3]{4})^3=6+6\sqrt[3]{2}+6\sqrt[3]{4}$; therefore
$\sqrt[3]{2}+\sqrt[3]{4}=\sqrt[3]{\{6(1+\sqrt[3]{2}+\sqrt[3]{4})\}}$, the answer.

4. Reduce $3-\sqrt{5}$, and $\sqrt{2}-2\sqrt{6}$ to a general surd.
$(3-\sqrt{5})^2=9-6\sqrt{5}+5=14-6\sqrt{5}$, and $\sqrt{(14-6\sqrt{5})}$, Ans.

6. Reduce $2\sqrt[3]{3}-3\sqrt[3]{9}$, and $7\sqrt[3]{3}-3\sqrt[3]{9}$ to a general surd.

In examples involving cube root radicals, it is useful to know the following form of the cube of a binomial: viz.

$$(a\pm b)^3=a^3\pm b^3+3ab(a\pm b)$$

Hence $(2\sqrt[3]{3}-3\sqrt[3]{9})^3=24-243-18\sqrt[3]{27}(2\sqrt[3]{3}-3\sqrt{9})=-219-54(2\sqrt[3]{3}-3\sqrt[3]{9})$; consequently

$\sqrt[3]{(-219-54(2\sqrt[3]{3}-3\sqrt[3]{9}))}$ is the general surd.

or, Ans. $\sqrt[3]{(162\sqrt[3]{9}-108\sqrt[3]{3}-219.)}$

$2\sqrt[3]{3}-3\sqrt[3]{9}$ multiply
$2\sqrt[3]{3}-3\sqrt[3]{9}$ by
$4\sqrt[3]{9}-6\sqrt[3]{27}$
$-6\sqrt[3]{27}+9\sqrt[3]{81}$
$4\sqrt[3]{9}-12\sqrt[3]{27}+9\sqrt[3]{81}$
or $4\sqrt[3]{9}-36+27\sqrt[3]{3}$
$2\sqrt[3]{3}-3\sqrt[3]{9}$
$8\sqrt[3]{27}-72\sqrt[3]{3}+54\sqrt[3]{9}$
$-12\sqrt[3]{81}+108\sqrt[3]{9}-81\sqrt[3]{27}$
$8\sqrt[3]{27}-108\sqrt[3]{3}+162\sqrt[3]{9}-81\sqrt[3]{27}$,
or $24-108\sqrt[3]{3}+162\sqrt[3]{9}-243$ Hence
I have $\sqrt[3]{\{(-219-54(2\sqrt[3]{3}-3\sqrt[5]{})\}}$
$-3\sqrt[3]{9}))$ Ans.

$49\sqrt[3]{9}-126+27\sqrt[3]{3}$ multiply
$7\sqrt[3]{3}-3\sqrt[3]{9}$ by
$343\sqrt[3]{27}-882\sqrt[3]{3}+189\sqrt[3]{9}$
$-147\sqrt[3]{81}+378\sqrt[4]{9}-81\sqrt[3]{27}$
$1029-1323\sqrt[3]{3}+567\sqrt[3]{9}-243$ or
$(786-1323\sqrt[3]{3}+567\sqrt[3]{9})^{\frac{1}{3}}$ Ans.

$7\sqrt[3]{3}-3\sqrt[3]{9}$ multiply
$7\sqrt[3]{3}-3\sqrt[3]{9}$ by
$49\sqrt[3]{9}-(21\sqrt[3]{27}=63)$
$-21\sqrt[3]{27}+9\sqrt[3]{81}$
Prod. $49\sqrt[3]{9}-126+27\sqrt[3]{3}$.

* $12\sqrt[3]{27}=12\times3=36$ and $9\sqrt[3]{81}=(3\times9)\sqrt[3]{3}=27\sqrt[3]{3}$, and $12\sqrt[3]{81}=12\sqrt[3]{(27\times3)}=36\sqrt[3]{3}$. Again $147\sqrt[3]{81}=147\sqrt[3]{(27\times3)}=147(\sqrt[3]{27}\times\sqrt[3]{3})=(147\times3)\times\sqrt[3]{3}=441\sqrt[3]{3}$, and $882\sqrt[3]{3}+441\sqrt[3]{3}=1323\sqrt[3]{3}$.

Case XI. *To extract the square root of a binomial or residual surd.*

Rule. Substitute the numbers, or letters, of which the given surd is composed, in the place of their equals, in one of the two following formulæ, according as it is a binomial or a residual, and it will give the root required.

$$\sqrt{(a+\sqrt{b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)})}+\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)})}$$
$$\sqrt{(a-\sqrt{b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)})}-\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)})}$$

And if the second part of the binomial, or residual, in this case, be an imaginary surd, the same theorems will still hold, by only changing $-b$ into $+b$, as below.

$$\sqrt{(a+\sqrt{-b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)})}+\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)})}$$
$$\sqrt{(a-\sqrt{-b})}=\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)})}-\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)})}$$

Where it is to be observed, that the only cases that are useful in this extraction, are when a is rational, and $a^2 - b$ in the first of these formulæ, or a^2+b in the latter, is a complete square.*

1. Find the square root of $11+6\sqrt{2}$, or $11+\sqrt{72}$.

Here $\sqrt{(\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\tfrac{11}{2}+\tfrac{1}{2}\sqrt{(121-72)})}=\sqrt{(\tfrac{11}{2}+\tfrac{7}{2})}=3$; and $\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)})}=\sqrt{(\tfrac{11}{2}-\tfrac{1}{2}\sqrt{(121-72)})}=\sqrt{(\tfrac{11}{2}-\tfrac{7}{2})}=\sqrt{2}$; whence $\sqrt{(11+6\sqrt{2})}$, or $\sqrt{(11+\sqrt{72})}=3+\sqrt{2}$, Ans. And by a similar substitution in the second formula, we shall have $\sqrt{(11-6\sqrt{2})}=3-\sqrt{2}$.

2. It is required to find the square root of $1+4\sqrt{-3}$, or $1+\sqrt{-48}$.

Here $\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+48)})}=\sqrt{(\tfrac{1}{2}+\tfrac{7}{2})}=2$; & $\sqrt{(\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2+b)})}=\sqrt{(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{(1+48)}}=\sqrt{(\tfrac{1}{2}-\tfrac{7}{2})}=\sqrt{-3}$; whence $\sqrt{(1+4\sqrt{-3})}$, or $\sqrt{(1+\sqrt{-48})}=2+\sqrt{-3}$, Ans.

And by operating in the same manner with the last formula, we shall have $\sqrt{(1-4\sqrt{-3})}=2-\sqrt{-3}$.

3. Required to find the square root of $6\pm2\sqrt{5}$, or $23 \pm 8\sqrt{7}$, or $36\pm10\sqrt{11}$, or $33\pm12\sqrt{6}$, or $((1+4\sqrt{-3})=(1+\sqrt{-48}))$, or $((3\pm4\sqrt{-1})=(3\pm\sqrt{-16}))$, or $-1+\sqrt{-8}$, or $a^2+2x\sqrt{(a^2-x^2)}$, or $6+2\sqrt{2}-\sqrt{(12)}-\sqrt{(24)}$, each of the 9 foregoing questions.

Here I have $\sqrt{(\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)})} = \sqrt{(\tfrac{6}{2} + \tfrac{1}{2}\sqrt{(36 - 20)})}=\sqrt{(3+2)}=5$, and $\sqrt{\tfrac{1}{2}}-\tfrac{1}{2}(a^2-b)=\sqrt{(\tfrac{6}{2}-\tfrac{1}{2}\sqrt{(36-20)})}=\sqrt{(3-2)}=1$, and $\therefore \sqrt{5}\pm1 =$ Ans. or $\sqrt{(\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{(a^2-b)}} = \sqrt{\tfrac{23}{2}} +$

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* This method of extracting the square root of a binomial surd, was known as early as the time of LUCAS DE BURGO; who, in his Treatise entitled *Summa de Arithmetica, &c.* printed in 1494, has given a rule for that purpose, which, though not expressed in general terms, is similar to the one above used. Rules for extracting the cube roots of certain binomial surds, which are useful in some cases of cubic equations, were also introduced, with the solutions of these equations, by TARTALIA and CARDAN, in their Treatises on this subject; and others, of a different kind, have since been invented by later writers; but all these rules, as well as those for the higher orders, are more or less tentative, and only answer for particular numbers; in which cases, that given in the text will be found as commodious as any that has yet been proposed.

$\frac{1}{2}\surd(529-448))=\surd(11\frac{1}{2}\pm4\frac{1}{2})=4$, and $\surd7$, $\therefore\ 4\pm\surd7$ the 2nd Ans., the 3d, $\surd(\frac{1}{2}a\pm\frac{1}{2}\surd(a^2-b)=\surd(\frac{36}{2}\pm\frac{1}{2}\surd(36^2-1100))=\surd(18\pm7)=5$, and $\surd11$; $\therefore\ 5\pm\surd11$, Ans.; or I have the 4th, $\surd(\frac{1}{2}a\pm\frac{1}{2}\surd(a^2-b)=\surd(\frac{33}{2}\pm\frac{1}{2}\surd33^2-864))=\surd(16\frac{1}{2}\pm7\frac{1}{2})=\surd24$ and 3; $\therefore\ 2\surd6\pm3$, Ans.

CASE XII. *To extract the cube root of a binomial, or residual surd.*

RULE. Let the surd, that is to have its root extracted, be of the form $\overset{3}{\surd}(a+\surd b)$, or $\overset{3}{\surd}(a-\surd b)$.

Then, if a^2-b be a perfect integral cube, and some whole number, can be found, that when substituted for n, will make $n^3-3\{\overset{3}{\surd}(a^2-b)\}n=2a$, the roots of two expressions, in this case, will be $\overset{3}{\surd}(a+\surd b)=\frac{1}{2}n+\frac{1}{2}\surd\{n^2-4\overset{3}{\surd}(a^2-b)\}$, and $\overset{3}{\surd}(a-\surd b)=\frac{1}{2}n-\frac{1}{2}\surd\{n^2-4\overset{3}{\surd}(a^2-b)\}$. And if the second part of the binomial, or residual, be an imaginary surd, and a^2+b be a perfect integral cube, the extraction may be effected, by finding the integral value of n in the following equation, as before, $n^3-3\{\overset{3}{\surd}(a^2+b)n\}=2a$. In which last case, the roots of the two expressions will be $\left\{\begin{array}{l}\overset{3}{\surd}(a+\surd-b)=\frac{1}{2}n+\frac{1}{2}\surd\{n^2-4\overset{3}{\surd}(a^2+b)\}\\ \overset{3}{\surd}(a-\surd-b)=\frac{1}{2}n-\frac{1}{2}\surd\{n^2-4\overset{3}{\surd}(a^2+b)\}\end{array}\right\}$, each of which latter formulæ may be obtained, by barely changing the sign of b in the former.

1. Find the cube root of $10\pm6\surd3$, or $10\pm\surd108$.

Here $a=10$, and $b=108$; whence $(\overset{3}{\surd}(a^2-b)=\overset{3}{\surd}(100-108)=-2$, and $n^3-3\{\overset{3}{\surd}(a^2-b)n\}=20$, or $n^3+6n=20$, where it readily appears, from inspection, that $n=2$. Whence $\overset{3}{\surd}(10+\surd108)=\frac{2}{2}+\frac{1}{2}\surd(4-4\times-2)=1+\frac{1}{2}\surd12=1+\surd3$, and $\overset{3}{\surd}(10-\surd108)=\frac{2}{2}-\frac{1}{2}\surd(4-4\times-2)=1-\frac{1}{2}\surd12=1-\surd3$.

For Trinomial, Quadrinomial Surds, &c.

RULE. Divide half the product of any two radicals by a third, gives the square of one radical part of the root; this repeated with different quantities, will give the squares of all the parts of the root, to be connected by + and —. But if any quantity occur oftener than once, it must be taken but once. For if $x+y+z$ be any trinomial surd, its square will be $x^2+y^2+z^2+2xy+2xz+2yz$; then if half the product of any two rectangles, as $2xy+2xz$, (or $2x^2yz$) be divided by some third $2yz$, the quotient $\frac{2x^2yz}{2yz}=x^2$, must needs be the square of one of the parts; and the like for the rest.

Extract the □ root of $10+\surd(24)+\surd(40)+\surd(60)$

Here $\frac{\surd(24\times\surd(40)}{2\surd(60)}=2$, and $\frac{\surd(24)\times\surd(60)}{2\surd(40)}=\surd9=3$, and $\frac{\surd(40)\times\surd(60)}{2\surd(24)}=\surd(25)=5$. And the root is $\surd2+\surd3+\surd5$.

Case XIII. *To extract any root* (*c*) *of a binomial surd $A \pm B$.*

Rule. Let $A^2 - B^2 = D$, take Q such, that $QD = n^3$, the last integer power. Let $\sqrt{\{(A+B) \times \sqrt{Q}\}} = r$, the nearest integer number. Reduce $A\sqrt{Q}$ to the simplest form $p\sqrt{s}$; let the nearest integer $\frac{r + \frac{n}{r}}{2\sqrt{s}} = t$; then the root $= \frac{t\sqrt{s} \pm \sqrt{(t^2 s - n)}}{\sqrt[2c]{Q}}$; if it can be extracted. Note. + is for the binomial $A + B$, and — for the residual $A - B$.

1. What is the cube root of $\sqrt{968} + 25$.

Here $D = 343 = 7 \times 7 \times 7$, and $Q \times 7^3 = n^3$; $Q = 1$, $n = 7$, then $\sqrt[3]{\{(A+B) \times \sqrt{Q}\}} = \sqrt[3]{56} + = r = 4$, and $A\sqrt{Q} = \sqrt{968} = 22\sqrt{2} = p\sqrt{s}$, and $\sqrt{s} = \sqrt{2} \cdot \frac{r + \frac{n}{r}}{2\sqrt{s}} = \frac{4 + \frac{7}{4}}{2\sqrt{2}} = t = 2$, and $t\sqrt{s} = = 2\sqrt{2}$; $\sqrt{(t^2 s - n)} = \sqrt{(8-7)} = 1$; $\sqrt[6]{Q} = 1$, and the root $\frac{2\sqrt{2}+1}{1} = 2\sqrt{2}+1$, which I find to succeed.

2. Find the cube root of $68 - \sqrt{4374}$.

Here $D = 250 = 5 \times 5 \times 5 \times 2$, and $5^3 \times 2^3 = 4D = QD = n^3$; and $Q = 4$, $n = 2 \times 5 = 10$, and $\sqrt[3]{\{(A+B) \times \sqrt{Q}\}} = \sqrt{(134 \times 2)} = 6 = r$; $A\sqrt{Q} = 136\sqrt{1} = p\sqrt{s}$, and $\sqrt{s} = 1$; $\frac{r + \frac{n}{r}}{2\sqrt{s}} = \frac{7\frac{2}{3}}{2} = \frac{23}{6} = 4 = t$, and $t\sqrt{s} = 4$, and the root $\frac{t\sqrt{s} - \sqrt{(t^2 s - n)}}{\sqrt[6]{2}} = \frac{\sqrt{(16-10)}}{\sqrt[6]{4}} = \frac{4 - \sqrt{6}}{\sqrt[3]{2}}$; for its cube is $68 - 27\sqrt{6}$.

Case XIV. *To find such a multiplier, or multipliers, as will make any binomial surd rational.*

Rule 1. When one, or both of the terms, are any even roots; multiply the given binomial, or residual, by the same expression, with the sign of one of its terms changed, and repeat the operation in the same way, as long as there are surds; when the last result will be rational.

In like manner, a trinomial surd may also be rendered rational, by changing the sign of one of its terms for the multiplier; and a quadrinomial surd by changing the signs of two of its terms, &c.

2. When the terms of the binomial surds are odd roots, the rule becomes more complicated; but for the sum, or difference, of two cube roots, which is one of the most useful cases, the multiplier will be a trinomial surd consisting of the squares of the two given terms, and their product, with its sign changed.*

Given surd	$5+\sqrt{3}$	$\sqrt{5}+\sqrt{3}$	$\sqrt[4]{5}+\sqrt[4]{3}$	given surd.
Multiplier	$5-\sqrt{3}$	$\sqrt{5}-\sqrt{3}$	$\sqrt[4]{5}-\sqrt[4]{3}$	1st mult.
	$25+5\sqrt{3}$	$5+\sqrt{15}$	$\sqrt{5}-\sqrt{3}$	1st prod.
	$-5\sqrt{3}-3$	$-\sqrt{15}-3$	$\sqrt{5}+\sqrt{3}$	2d mult.

a

Product $25-3=22$ $5-3=2$ $5-3=2$ Ans. ratio.

$\sqrt{5}+\overline{\sqrt{3}-\sqrt{2}}$ given surd.
$\sqrt{5}+\sqrt{3}+\sqrt{2}$ 1st mult.
$5+\sqrt{15}-\sqrt{10}$
$+\sqrt{15}+3-\sqrt{6}$
$+\sqrt{10}+\sqrt{6}-2$
$6+2\sqrt{15}$ 1st product.
$-6+2\sqrt{15}$ 2d multiplier.
$-36-12\sqrt{15}$
$+12\sqrt{15}+60$
$60-36=24$, Ans.

$\overline{\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}}$ mult.
$\sqrt[3]{7}-\sqrt[3]{5}$ given surd.
$\sqrt[3]{343}+\sqrt[3]{245}+\sqrt[3]{175}$
$-\sqrt[3]{245}-\sqrt[3]{175}-\sqrt{125}$
$\sqrt[3]{343}$ * * $\sqrt[3]{325}=7-5$
Given $\sqrt[3]{7}+\sqrt[3]{3}$ surd.
Mult. $\sqrt[3]{7^2}-\sqrt[3]{21}+\sqrt[3]{3^2}$
$7-\sqrt[3]{147}+\sqrt[3]{63}$
$+\sqrt[3]{147}-\sqrt[3]{63}+3$
Ans. $10=7+$ * * 3

CASE XV. *To reduce a fraction, whose denominator is a surd, to another that shall have a rational denominator.*

RULE 1. When the proposed fraction is a simple one, of the form $\frac{b}{\sqrt{a}}$, multiply each of its terms by $\sqrt{a}$, and the resulting fraction will be $\frac{b\sqrt{a}}{a}$. Or when it is of the form $\frac{b}{\sqrt[3]{a}}$, multiply the terms by $\sqrt[3]{a^2}$, and the result will be $\frac{b\sqrt[3]{a^2}}{a}$. And for the general form $\frac{b}{\sqrt[n]{a}}$, multiply by $\sqrt[n]{a^{n-1}}$, & the result will be $\frac{b\sqrt[n]{a^{n-1}}}{a}$

2. If it be a compound surd, find such a multiplier, by one of the rules of the last case, as will make the denominator rational; and multiply both the numerator and denominator by it.

1. Reduce the fractions $\frac{2}{\sqrt{3}}$; $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$, and $\frac{3}{\sqrt{5}-\sqrt{2}}$, to a fractions whose denominators shall be rational.

Here $\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$; and $\frac{3}{\frac{1}{2}\sqrt[4]{5}}\times\frac{\sqrt[4]{5^3}}{\sqrt[4]{5^3}}=\frac{3\sqrt[4]{5^3}}{\frac{1}{2}\times5}$ $=\frac{6\sqrt[4]{5^3}}{5}=\frac{6}{5}\sqrt[4]{125}$. Here $\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=$ $\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=\frac{\sqrt{5}+\sqrt{2}}{1}=\sqrt{5}+\sqrt{2}$, Ans.

Cor. A binomial becomes rational after one operation, a trinomial after two, and a quadrinomial after three, &c.

Cor. The number of operations is equal to the power of 2 in the index.

A and B began to trade with equal sums of money. In the first year A gained 40 dollars and B lost 40; but in the second A lost one third of what he then had, and B gained a sum less by 40 dollars than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with? Ans. \$320.

Cubic Equations.

Let $dx^3 + cx^2 + bx = N$ be any cubic equation, and let, as in the solution of quadratics $x = r + y$, or $y + r = x$, then

$$\left.\begin{array}{r} by + br = bx \\ cy^2 + 2cry + cr^2 = cx^2 \\ dy^3 + 3dry^2 + 3dr^2y + dr^3 = dx^3 \end{array}\right\} = N$$

$$dy^3 + c'y^2 + b'y + A = N \quad \text{Sum.}$$

Or, $dy^3 + c'y^2 + b'y = (N - A) = N'$

Again, $y = s + z$, then

$$\left.\begin{array}{r} b'z + b's = b'y \\ c'z^2 + 2c'sz + c's^2 = c'y^2 \\ dz^3 + 3dsz^2 + 3ds^2z + ds^2 = dy^3 \end{array}\right\} = N$$

$$dz^3 + c''z^2 + b''z + A' = N' \quad \text{Sum.}$$

$$\left\{\begin{array}{c} \text{Given } bx^2 + ax = n \\ b \quad a \\ br + a \\ 2br + a = a' \\ b \quad a' \\ bs + a' \\ 2bs + a' = a'' \end{array}\right.$$

Or $dz^3 + c''z^2 + b''z = (N' - A') = N'$ Here it is evident that these equations are similar to the first,and thus we may successively find the values of r, s, t, &c. &c. the figures of the root. Now to find the denominators, put down the coefficients in a line thus,

$$\begin{array}{llll} d \quad c & b & d \quad c' & b' \\ dr + c, & dr^2 + cr + b & ds + c', & ds^2 + c's + b', \\ 2dr + c, & 3dr^2 + 2cr + b = b' & 2ds + c', & 3ds^2 + 2c's + b', \\ 3dr + c, = c'. & & 3ds + c'. & \\ \&c. & \&c. & & \&c. \end{array}$$

And if we suppose $d = 1$, then the equation will be $x^3 + cx^2 + bx = n$. And then the values of r, s, t, &c. are as follows:

$$\frac{N}{r^2 + cr + b} = r; \quad \frac{N'}{s^2 + c's + b'} = s; \quad \frac{N''}{t^2 + c''t + b''} = t, \text{ \&c. \&c.}$$

Now, by substituting the values of c', b', c'', b'', &c. we have

$$1. \quad \frac{N}{r^2 + cr + b} = \frac{N}{r(r + c) + b} = r.$$

$$2. \quad \frac{N'}{s^2 + (b + 3r)s + 3r + 2br + b} = \frac{N''}{s(s + 3r + c) + 3r^2 + 2cr + b} = s.$$

$$3. \quad \frac{N''}{t^2 + c''t + b''} = \frac{N''}{t(t + c + 3(r + s)) + 3s^2 + 2c's + b} = t.$$

$$4. \quad \frac{N'''}{u^2 + c'''u + b'''} = \frac{N'''}{u\{u + c + 3(r + s + t)\} + 3t^2 + 2c''t + b''} = u.$$

Hence it is evident how these formulas may be continued at pleasure, and all the quantities contained in the same are known; and since, in the second formula, $3r^2 + 2cr + b$ form apart of the divisor, this may be used as a trial divisor for finding s, and so for the rest. An example will render these formulas very easy.

Given $x^3+9x^2+4x=80$, to find x.

Here 2 is the first figure of the root found by trial.

$$c=9,\ b=4,\ N=80,\ r=2.$$

$$\begin{array}{lrlr}
 & b=4 & c=9.\quad \text{Root} & \\
 & \overline{r(r+c)=22} & 80(2.47213596 & \\
\text{First divisor} & =r^2+cr+b=26 & 52\ r.stu\ \text{\&c.} & \\
 & r^2=4. & \overline{28=N'} & \\
s^2+c's & =s(s+3r+b)=6.16 & 23.264 & \\
\text{2d divisor is} & =\overline{s^2+c's+b'=58.16} & \overline{4.736=N''} & \\
 & s^2=.16 & 4.593323 & \\
 & t(t+c+3(r+s)=1.1389 & \overline{.142677} & *398 \\
\text{3d divisor is} & \overline{t^2+c''t+b''=65.6189} & .133591 & 334 \\
\text{And so on,} & 49 & \overline{9086} & \overline{64} \\
(9+(2.47\times3)+.002)2 & =16\cdot4|12\quad .032824 & 6683 & 60 \\
\text{Fourth divisor is} & \overline{66.795|5} & \overline{2403} & \overline{4} \\
16.3161\times1 & \cdots\cdots 1|6\cdot4161\ \ldots\ 2 & 2005 & 4 \\
\text{Fifth divsior is} & \ldots\ \overline{|6|6\cdot|8|3|0} & \overline{*398} &
\end{array}$$

Hence, from these formulas, we derive the following rule. Put down b the coefficient of x, and a little to the right place the absolute number, which is to be considered as a dividend, the figures of the root forming the quotient. Place the first figure of the root found by trial in the quotient, above which write the coefficient of x^2, observing that its unit's place be over the unit's place of the quotient. Multiply the value of the quotient figure, taking in those above by that value; add the product to b, and the sum is the first divisor. Write the square of the quotient figure just found under the first divisor. Find now the next figure of the root, and to its value (including those above it) prefix three times the preceding, taking in the value of the figure above it. Multiply the result by the last found figure, add the product to the three sums immediately above, and we shall have the *second* divisor; and in the same manner are the succeeding divisors to be obtained.

Given $x^3+4x^2+2x=2328$, to find the value of x.

Here the highest denomination of the root found by trial is 10.

$$\begin{array}{lrrl}
 & b & & 4=c \\
 & 2 & 2328\,(& 12\ \text{root.} \\
14\ldots. & 140 & 142 & \\
\text{Divisor} & \overline{142} & \overline{908} & \\
 & 100 & 908 & \\
36\ldots. & 72 & & \\
\text{Divisor} & \overline{454} & &
\end{array}$$

Here it must be observed, as in quadratics that the 1 in the quotient must be considered 10 thus, $(r+a)r=(10+4)\times10=140$; and so if were in the hundred's place it must be considered as 100, &c.

Biquadratic Equations.

For the solution of equations of a higher order, the principle is nearly the same. Let $Ex^4+Dx^3+Cx^2+Bx=N$ be any biquadratic equation; let r be the first figure in the root; then we shall have $\dfrac{N}{Er^3+Dr^2+Cr+B}=r$. Now let y denote the remaining figures of the root, then $y+r=x$, and by substitution,

$$\begin{array}{rcrcrcrcrcl}
 & & & & & & By & + & Br & = & Bx\\
 & & & & Cy^2 & + & 2Cry & + & Cr^2 & = & Cx^2\\
 & & Dy^3 & + & 3Dry^2 & + & 3Dr^2y & + & Dr^3 & = & Dx^3\\
Ey^4 & + & 4Ery^3 & + & 6Er^2y^2 & + & 4Er^3y & + & Er^4 & = & Ex^4\\ \hline
Ey^4 & + & D'y^3 & + & C'y^2 & + & B'y & + & A & = & N.
\end{array}$$

Or, $Ey^4+D'y^3+C'y^2+B'y=N'$, an equation similar to the one proposed; and if S be the first figure of the root, then we shall have $\dfrac{N'}{ES^3+D'S^2+C'S+B'}=S$. Now to find the denominators, put down the coefficients in a line, thus;

$$\begin{array}{llll}
E & D & C & B\\
 & Er+D, & Er^2+Dr+C, & Er^3+Dr^2+Cr+B\\
 & 2Er+D, & 3Er^2+2Dr+C, & 4Er^3+3Dr^2+2Cr+B=B'\\
 & 3Er+D, & 6Er^2+3Dr+C=C' & \\
 & 4Er+D=D', & & \\
E' & D' & C' & B'\\
 & Es+D', & Es^2+D's+C', & Es^3+D's^2+C's+B'\\
 & 2Es+D', & 3Es^2+2D's+C', & 4Es^3+3D's^2+2C's+B'\\
 & 3Es+D', & 6Es^2+3D's+C', & \\
 & 4Es+D, & \&c. \qquad \&c. & \&c.
\end{array}$$

Hence it is evident how these lines are formed, viz. by multiplying the succeeding terms by r, s, &c. and adding them respectively. By this means figurative numbers of the 2nd, 3d, &c. degrees are formed; (see Simpson's Alg.) and so we may proceed to an equation of any degree. Thus, let there be given the equation, $Fx^5+Ex^4+Dx^3+Cx^2+Bx=N$. Let r be the first figure of the root. Then $r+y=x$, or $y+r=x$.

$$\begin{array}{rcrcrcrcrcrcl}
 & & & & & & & & By & + & Br & = & Bx\\
 & & & & & & Cy^2 & + & 2Cry & + & Cr^2 & = & Cx^2\\
 & & & & Dy^3 & + & 3Dry^2 & + & 3Dr^2y & + & Dr^3 & = & Dx^3\\
 & & Ey^4 & + & 4Ery^3 & + & 6Er^2y^2 & + & 4Er^3y & + & Er^4 & = & Ex^4\\
Fy^5 & + & 5Fry^4 & + & 10Fr^2y^3 & + & 10Fr^3y^2 & + & 5Fr^4y & + & Fr^5 & = & Fx^5\\ \hline
Fy^5 & + & E'y^4 & + & D'y^3 & + & C'y^2 & + & B'y & + & A & = & N
\end{array}$$

Or, $Fy^5+E'y^4+D'y^3+C'y^2+B'y=N'$, an equation similar to the proposed one whose root s must be such, that if N' be divided by $Fs^4+E's^3+D's^2+C's+B'$, the quotient will be s. Now, to find the denominator of these fractions, multiply by r, s, &c., as before. Thus,

$$\begin{array}{lllll}
F \quad E & D & C & B & \\
\quad Fr+E, & Fr^2+Er+D, & Fr^3+Er^2+Dr+C; & Fr^4+Er^3+Dr^2+Cr+B & \\
2Fr+E, & 3Fr^2+2Er+D, & 4Fr^3+3Er^2+2Dr+C; & 5Fr^4+4Er^3+3Dr^2+2Cr+B=B' & \\
3Fr+E, & 6Fr^2+3Er+D, & 10Fr^3+6Er^2+3Dr+C=C' & & \\
4Fr+E, & 10Fr^2+4Er+D,=D' & & & \\
5Fr+E=E'. & & & &
\end{array}$$

$$\begin{array}{lllll}
F \quad E' & D' & C' & B' & \\
\quad Fs+E' & Fs^2+E's+D' & Fs^3+E's^2+D's+C' & Fs^4+E's^3+D's^2+C's+B' & \\
2Fs+E' & 3Fs^2+2E's+D' & 4Fs^3+3E's^2+2D's+C' & 5Fs^4+4E's^3+3D's^2+2C's+B'=B' & \\
3Fs+E' & 6Fs^2+3E's+D' & 10Fs^3+6E's^2+3D's+C' & & \\
4Fs+E' & 10Fs^2+4E's+D' & & & \\
5Fs+E', & \&c. & \&c. & \&c. &
\end{array}$$

From these principles being found to hold good in all equations whatever, we can determine the following general rule.. RULE. Arrange the coefficients of the given equation in a row, commencing with that of the first term.

Add the product of the first root figure, found by trial, and the first coefficient to the second coefficient; the product of the sum and same figure to the third coefficient, and so on to the last coefficient, and the last sum will be the divisor. Repeat this process with the first coefficient and these sums, and the number under the last sum will be the trial divisor for the next figure. Perform a similar process with the first coefficient and these second sums, stopping under the $n-1$th coefficient. Perform again a similar process with the same first coefficient and these last sums, stopping here under the preceding, or $n-2$th coefficient, and so on till the last sum falls under the second coefficient. Find now from the trial divisor the next figure of the root, and proceed with the last set of sums, and this new figure exactly the same as with the original coefficients and the first figure in finding the preceding divisor, and the next divisor will be obtained; and in a similar manner are the other divisors to be determined. Lastly, let us take the general equation. Let $Bx^n+\ldots Ex^4+Dx^3+Cx^2+Bx=N$ be an equation of any degree whatever, and let r be such a number in either of the series 0, 1, 2, 3, &c.; or series 0, 1, 2, 3, ; or 10, 20, 30, &c. ; that when it and the next succeeding number are separately substituted for x in the equation, the results shall be the one less, and the other greater than N; then r will be the first figure of one of the roots of the equation ; and if N be divided by $Rr^{n-1}+\ldots Er^3+Dr^2+Cr+B$, the quotient must be r. Suppose such a value of r is found, and let y represent the succeeding figures of the root ; then $y+r=x$, and therefore

$$\begin{array}{rrrrrl}
 & & & By & +\ Br & = Bx, \\
 & & Cy^2 & +\ 2Cry & +\ Cr^2 & = Cx^2, \\
 & Dy^3 & +\ 3Dry^2 & +\ 3Dr^2y & +\ Dr^3 & = Dx^3, \\
Ey^4 & +\ 4Ery^3 & +\ 6Er^2y^2 & +\ 4Er^3y & +\ Er^4 & = Ex^4, \\
\multicolumn{6}{l}{Ry^n+\ldots\ldots} \\
\hline
\multicolumn{6}{l}{Ry^n+\ldots\ldots E'y^4+D'y^3+C'y^2+B'y+A=N;}
\end{array}$$

and, by transposition,

$$Ry^n+\ldots\ldots E'y^4+D'y^3+C'y^2+B'y=N-A=N',$$

an equation similar to the one proposed, the first figure s, in the root y of which must be such, that if N' be divided by $Rs^{n-1}+\ldots\ldots E's^3+D's^2+C's+B'$, the quotient will be s; if, therefore, we suppose s to be found, and if z be put for the remaining figures of the root, we shall, by proceeding as before, get another equation $Rz^n+E''z^3+C''z^2+B''z=N''$; also similar to the first ; and if we continue this process, we may obtain, one by one, all the figures of the root x, and it is evident that each divisor will be similarly formed from the coefficients of the corresponding equation, and the new figure of the root.

Now it is obvious that $B'x$ is the nth term in the equation $Ry^n+\ldots\ldots E'y^4+D'y^3+C'y^2+B'y=N'$, or that the column represented by B' is the nth column from the left, and that it consists of n terms ; the column represented by C' is the $n-1$th, and consists of $n-1$ terms, &c. ; also each of these columns, omitting the numeral parts, is equal to the preceding multiplied by r, plus the corresponding coefficient in the proposed equation ; consequently, since the first column is simply R, if the second, third, &c. coefficients of the proposed equation be Q, P, &c. respectively, then the first, second, third, &c. columns, without the numeral coefficients, will be R, $Rr+Q$, Rr^2+Qr+P, &c. ; but if the first term in this series be multiplied by r, and the product added to the second, and the result be multiplied by r, and the product added to the third &c. &c. that the proper numeral coefficients will be obtained, because the above columns of numeral coefficients are the same as the several binomial columns, except that the above are written in a reverse order, that is, from right to left, and in this reverse order they will be produced by adding the series, as above. See Emerson Algebra Ed. 1764 page 287.

Given $x^3+90x^2+2700x=21168$, to find x.

```
                +90
     2700       21168(6.384761
96....576       19656        [967
Div. 3276       1512
       36       1176.147
108.3..32.49     335.853
Div. 3920:49     316.943072
            9     18.909928
108.98..8.7184    15.883799
Div. 3961.7884     3.026129
            64     2.780024
10|9.1|44..4366     246105
Div. 3970.9498      238292
            77  267*  7813
Div. 3971.463   288   3972
Div. 3|9|7|1.|54 29   3841
                 28   3574*
```

Given $x^3+2x^2-3x=9$, to find x.

```
                +2
  —3            9(1.9394650585
 3..3           8.379
Div. 0           .621
     1           469857
5.9...5.31       151143
Div. 9.31        143684019
      81           7458981
7.73...2319        6415316
Div.15.6619        1043665
          9         962514
7.799..70191         81151
Div.15.964891        80212
           81          939
7.8|17...313           802
Div.16.03829           137
          5.           128
  16.0419                9
1,6.0,4,2,4              8
```

Given $x^3+x^2=500$, to find x to 10 or 12 places.

```
                +1
          0     500(7.61727975
8.......56      392          [594
Div.     56     108
         49     104.736
22.6.13.56        3.264
Div.174.56        1.887181
         36       1.376819
23.81..23.81      1.323862113
Div.188.7181        52956887
           1        37858967
23.837..166859      15097920
Div.189.123159      13251090
              49     1846830
23|85|12..4770       1703729
Div.189.294837  1124  143101
           167   946  132512
Div.189.30128    178   10589
             2   170    9465
Div.1,8,9.3,0,3,2
```

Given $x^3+10x^3+5x=260$, to find x.

```
                +10
          5     260(4.11798686
14.....56       244
Div.     61      16
         16      13.521
22.1....2.21      2.479
Div.   135.21     1.376531
            1     1.102469
22.31.....2231      966091
Div.   137.6531     136378
              1     124372
2'2.3|37...1570  120 12006
Div.   138.013   111 11057
            21     9   949
       138.191     8   829
     1,3,8,.2,1        120
```

A and B between them owe \$240. A pays eight dollars a day, and B pays the first day \$1, the second 2, the third 3, and so on. In how many days will they clear the debt, and how much did each of them owe? Ans. 15 days.

2. Given $x^3 - 12x = -12$, to find x.

	—12	—12(1.115749397
	1	—11
	—11	—1
	1	—.869
3.1....	.31	—131
Div.	—8.69	— 83369
	1	—47631
3.31.....	331	—41435125
Div.	—8.3369	—6195875
	1	—5787588
3.335......	16675	—408287
Div.	—8.287025	—330620
	25	—77667
3.3457....	2342	—74388
	—8.267983	—3279
	13	—2479
	—8.26551	—800
	—8.2,6,5,3	—744
		—56
		—56

3. Given $x^3 + x^2 = 500$, to find x.

Here C=1, B=0 N=500, and the first figure of the root is 7.

	B=0	C=1.
8....	56	500(7.61728
D.	56	392
	49	108
22.6.	13.56	104.736
	174.56	3.264
	36	1.887
23·81..	24	1.377
	188.7.2	1.324
	1,8,8.9,6	53
		38
		15
		15

4 Given $x^3 + x^2 + x = 100$, to find x.

	1	1.
5....	20	100(4.264429978
Div.	21	84 Root.
	16	16
1.32···	2.64	11.928
Div.	59.64	4.072
	04	3.788376
13.66	··· 8196	283624
Div.	63.1396	256072
	36	27552
13.784	.. 55136	25631
Div.	64.017936	1921
	16	1282
13.792....	5517	639
	64.078605	577
	276	62
	64.078881	58
	1	4
	6,4.0,7,9	4

4. Given $x^3 - 18\frac{1}{12}x = -29\frac{53}{108}$, to find x. Or $x^3 - 18.08\dot{3}x = -29.4\dot{9}07\dot{4}$

```
                          . . 0.Root
     —18.083       —29.49074(2.3333
2.....  4
                   —28.166    [33333
Div. —14.083
        4          —1.324074
6.3.... 1.89       —1.2579

Div.  —4.193          —66074
          9            60163
6.93....  2079
                     —5911074
Div.  —2.00543       —5326962
             9        —584112
6.993....  20979      —525770
Div.  —1.775654        —58342
              9        —52507
6.99|93.... 210         —5835
     —1.75256           —5251
            2      6     —584
     —1.7,5,0,2    5     —525
                   1      —59
                          —53
```

The root is 2.333333333, true to 10 places, evidently shews that the true root is 2.3 or 2⅓ exactly. The root is carried so far to show that the 3's are all repeaters.

5. Given $x^3 - 2x = 2$, to find x.

```
   —2     2(1.7693
1....1   —1
D.   1    3
     1    2.513
3.7..2.59  .487
D.  3.59   .419
      49     68
5.16..31     66
D.  6.98      2
    7.3       2
```

Ans. 1.7693.

6. Given $x^3 - 9x = 5$, to find x.

```
   —9    +0Root
3....9   5(3.24655
D.   0   0
     9   5
9.2..1.84 3.968
D. 19.84  1.032
        4  .8844
9.64...39  1476
D.  22.11  1353
        5   123
    22.55   112
             11
             11
```

7. Given $x^2 + 1728\frac{1}{3}x = 123578$, or $x^2 + 1728.\dot{3}x = 123578$ to find the two roots or values of x.

Ans. 68.7652857 or —1797.0986190.

8. Given $x^2 - \frac{127}{364}x = \frac{2547}{59825}$, or $364x^2 - 127x = \frac{927108}{59825} =$ 15.49699958, to find the two values of x.

Ans. $x. =.4446489374$ or —.0957478385.

9. Given $x^2 - \frac{101}{102}x = -\frac{19003}{119072}$, or $102x^2 - 101x = \frac{193830}{119072}$ $= -16.27843657$, to find the two values of x.

Ans. $x=.2026436785$ or .787552399.

```
1728.3   123578(68.7652857
  60      10730   (1797.0986190
1788.3·   16278
  60     14850.6·
1848.3·   1427.3·
   8     1305.523.
1856.3·   121.810·
   8      111.9476
1864.3·     9.8624
  .7        9.3293
1865.03      5331
   7         3732
1865.73·     1599
    6        1493
1865.793·     106
    6          93
1865.853·      13
     5         13
1,8,6,5.8,5,8
```

Those figures which are followed by inverted periods, are to represent repeaters.

Question 2.9.

```
                1,0,2 —101— 16.27&c. (.2026
                      20.4—16.12 (436785
1st divis.      —80.6      —158436
                 20.4      —119992
trial do.       —60.2     —3844457
                 .204     —3583848
complete do —   59.996    —260609
                  204     —238662
trial do.   —   59.792     —21947
                  612      —17898
complete do. —  59.7308     —4049
                  612       —3580
trial do. —     59.6696      —469
                   41        —418
complete do. —  59.665,5      —51
                    4         —48
              —5,9.6,6,1       —3
Two roots, .2026436785        — 3
           .7875523999
```

```
3.6.4 —127 15.49(.444648937)
 145.6  7.44 —·0957478
  18.6  8.0569
 145.6  7.1504
 164.2   .906599
 14.56   .789104
178.76  12749558
 14.56  11787024
193.32    962533
 1.456    786734
194.776   175800
 1.456    157361
196.232      736
  2184       590
196.4504     146
  2184       138
196.6688       8
   146         8
196.683,4  *196.698
    15           3
*196.698   1,9,6.7,0,1
```

16

11

10. Given $x^3-6x=5.6$, to find x.

	−6	0.Root.
2....	4	5.6(2.8252638
Div.	−2	−4
	4	9.6
6.8...	5.44	9.152
Div.	11.44	.448
	64	,353768
8.42..	1684	94232
Div.	17,6884	89498
	4	4734
8.4,6,5,,,	423	3589
	17.8995	1145
	2	1077
	1,7.9,4,4	68
		54
		14
		14

11. Given $x^3-12x=15$, to find x.

	−12		0.	
3....	9		15(3.971960	
Div.	−3		−9	[768
	9		24	
9.9...	8.91		21.519	
Div.	23.91		2.481	
	81		2.411773	
11.77...	8239		69227	
Div.	34.4539		35295	
	49		33932	
11.911...	119		31786	
	35.2946	3	2146	
	11	2	2119	
	3,5.3,1,7	1	27	
			24	
			3	

12. Given $x^3+9x=6$, to find x.

	9	0
6....	.36	6(.6378342
Div.	9.36	5.616
	36	.384
1.83....	549	.304047
Div.	10.1349	79953
	9	71428
1.897....	133	8525
Div.	10.2040	8175
	2	350
	10.219	307
	1,0.2,2	43
		41
		2
		2

13. Given $x^3-3x=3$, to find x.

	−3	0. Root.
2....	4	3(2.1038033531
Div.	1	2
	4	1·
6.1 .	.61	·961
Div.	9.61	39
	1	30746727
6.303..	18909	8253273
Div	10.248909	8218805
	9	34468
6.3098..	5679	30838
	10.273506	3630
	1,0.2,7,9,195	3084
		546
		514

14. Given $x^3 - 6x = 2$, to find x.

—6	0. Root.	
2....4	2(2.601679	
D. —2	—4 [1319	
4	6	
6.6...3.96	5.976	
D. 9.96	24	
36	14287801	
7.801. 7801	9712199	
14.287801	8580171	
1	1132028	
4682	1001386	
14.300285	130642	
55	128755	
14.3055,2	27	1887
1,4.3,0,6,1	14	1431
	13	456
	13	429

15. Given $x^3 + .75x^2 + .875x = .5625$, to find x.

.875	+.75
1.15.. 46	.5625 (.41437
D. 1.335	.534 0504
16	285
1.96.. 196	19746
D. 1.9746	8754
1	8008944
1.984...7936	745056
D. 2.002236	603236
16	141820
1.9923.. 598	140806
2.010786	1014
14	1006
2.01152	8
2.0,1,1,7	8

16. Given $x^3 + \frac{8}{9}x^2 - 5x = -\frac{8}{3}$, to find x.

	$.\dot{8}$
—5	$-2.\dot{6}$(1.3510298
$-1.\dot{8}$	[502
	$-3.\dot{1}$
$-3.\dot{1}$	$.\dot{4}$
1	$.310\dot{3}$
$4.1\dot{8}$..$1.25\dot{6}$	
$1{\cdot}03\dot{4}$	$134\dot{1}$
9	$131152\dot{7}$
$4.83\dot{8}$..$2419\dot{4}$	$2958\dot{3}$
$2{\cdot}6230\dot{5}$	$2872439\dot{8}$
25	
49348.. $4934\dot{8}$	$85893\dot{4}$
	575496
$2.872439\dot{8}$	283438
1	258973
10	24465
2.8,7,7,4,8	23020
	1445
	1438
	6—7

17. Given $x^3 - 24x^2 = -528.880073$ to find x.

0	—528.8800735
—19.—95	—475.
—95	—53.880
25	—50.283
—8.7.—2.61	—3.597073
—167.61	—3.405832
9	—191241
—8.08..—1616	—170461
—170.2916	—20780
4	—17047
—8,0.39...—80	—3733
—170.460,9	—3409
— 8	—324
—1,7,0.4,7,0	—171
	—153
24.	—153

$x = 5.3211219$ Ans.

Given $y^3 - 3.4y^2 - 3y = -.6$, to find y to 16 places.

```
                              —3.4
              — 3             —.6(.1691770039038507
— 3.3......— .33          — .333              root.
Diviosr    —3.33          — 267
               1          — 229944
— 3.04.....  —1824        — 37056
Divisor.   —3.8324        — 36336591
               36           — 719409
— 2.911..........26199      — 406380629
Divisor.   — 4.037399       — 313028371
                 81         — 284560870787
— 2.8929........—28929        — 28467500213
Divisor.   — 4.06380629       — 28451646935
                   1             — 15853278
— 2.89263........2025841         — 12193624
Divisor.   — 4.0642981541         — 3659654
                     49           — 3658087
—2.892843..........202574          — 1567
Divisor.   —4.064520990            — 1219
        — 4.|0|6|4|5|4|1|247        — 348
                                    — 325
                                     — 23
                                     — 20
                                        3
```

Given $x^3 - 240x^2 + 19200x = 61115.5$, to find x.

```
                        —240          root.
           19200       61115.5(3.31888216121
— 237. . . —711        55467
Divisor    18489        5648.5
add            9        5315.337
—230.7     — 69.21       333.163
Divisor    1771.79       176.463691
add              9       156.699309
—230.09    — 2.3009      141 378224
Divisor    17646.3691     15.5614866
                    1     14.1121626
—230.0|62  — 1.8405         14493240
Divisor    17642.227|8      14112000
—23|0.0452      —1840         381240
Divisor    17640.20|33        352800
—|230.04352     —184           28439
Divisor    17640.0|009         17640
                   — 4         10800
Divisor    1.76|3|9.|9|8       10584
  Rem. 216—176=40—35=4.          215
```

Given $52x^4 -$
753144, to

```
52
   —441584
     52000
   —389584
     52000
   —337584
     52000
   —285584
     52000
   —233584
     20800
   —212784
     20800
   —191984
     20800
   —171184
     20800
   —150384
      2600
   —147784
      2600
   —145184
      2600
   —142884
      2600
   —139984
       416
   —139568
       416
   —139152
       416
   —138736
       416
   —138320
```

$441584x^3-771997102.9985x^2+763080806290.03x=1620020551$ find x to 18 or 20 places of decimals.

Root, 1458.0156932762799496.

771997102.9985	763080806290.03	162002055 753144.0000
– 389584000	382413102998.50	1145393909288530.0000
382413102.9985	1145493909288.53	4745266424464614.0000
– 337584000	44829102998.50	423990245394572.0000
44829102.9985	1190323012287.03	50536397070042.0000
– 285584000	130347308800.60	43749226491787 7500
– 240754897.0015	1059975613486.43	6787170578254.2500
85113600	161064838800.6000	6773044690061.3440
– 325868497.0015	898910774685.8300	13225888192.9060
76793600	23926244850.0750	8427769782.58374
402662097.0015	874984529835.7550	4798118410.32226
68473600	24289204850.0750	4213848172.06165
— 471135697.0015	850695324985.6800	584270238.26061
7389200	33952238728.0120	505660113.21862
— 478524897.0015	846743086257.6680	78610125.04199
7259200	3961144456.0120	75848986.16508
— 485784097.0015	842781941801.6560	2761138.87691
7129200	4962543.2821	2528299.40037
— 492913297.0015	842776978258.3739	232839.47654
— 1116544	4962557 114	168553.29304
— 494029841.0015	842772015701.059	64286.18350
1113216	2481288.030	58993.65256
— 49543057.0015	842769634412.32|9	5292.53094
1109888	— 2481292.39	5056.59879
— 496252945.0015	842767153119.93	235.93215
383.208	297755.58	168.55329
— 496254428.21	842766855364.3|8	67.37886
1383.2	— 297755.6	58.99365
— 496255711.4	842766559608.7	8.38521
1383.2	— 44663.3	7.58490
496257094.6	842766512945.|4	.80031
691.6	— 44663.	75849
496257786	84276646828|2.	4182
692	— 149	3371
— 49625847|8	84276646679	811
69	49	759
— 49625917	84276646530	52
— 8	10	52
— 4|9|6|2|5|9|2|5	8|4|2|7|6|6|4|6|5|2|0	

Extract the root of the equation $x^5+6x^4-10x^3-112x^2-207x=110$. Here the first figure of the root is 4.

1	6	—10	—112	—207	110(4.464101
	4	40	120	32	—700
	10	30	8	—175	810
	4	56	344	1408	667.05984
	14	86	352	1233	142.94016
	4	72	632	434.6496	133.46395
	18	158	984	1667.6496	9.47621
	4	88	102.624	477.4144	9.24089
	22	246	1086.624	2145.0640	23532
	4	10.56	106.912	79.3352	23158
	26.4	256.56	1193.536	2224.39912	374
	.4	10.72	111.264	80.389	232
	26.8	267.28	1304.800	2304.788	142
	.4	10.88	17.454	5.434	139
	27.2	278.16	1322.2513	2310.22|2	3
	.4	11.04	17.56	5.44	2
	27.6	289.20	1339.811	2315.66	
	.4	1.68	17.6	.14	
	2|8.|0	290.8|8	1357.4	2315.8|0	*1358.
		1.7	1.|2	.1	1
		2|9|2.|6	1358.|6*	2|2|3|1|5.9	1|3|6

Give $x^5+6x^4-10x^3-112x^2-207x=-110$ find the value of

1	26.3	246	984	1233	—110(4.363261
	3	7.89	76.167	318.0501	—700 ro
	26.6	253.89	1060.167	1551.0501	+590
	.3	7.98	78.561	341.6184	465.31503
	26.9	261.87	1138.728	1892.6685	124.68497
	3	8.07	80.982	74-1897	118.01149
	27.2	269.94	1219.701	1966.858,2	6.67348
	.3	8.16	16.785	7.5202	6.13762
	27.5	278.10	1236.49,5	2042.060	53586
		1.65	16.88	3.814	40998
		279.75	1253.3,7	2945.87,3	12588
		1.7	17.0	3.81	12301
		281.4	1270.3	2049.68	287
		2.	.8	.25 this	205
		28,3	1,2,7,1	2049.93 add	82
				2,0,5.,0,2 sum	82

22. Given $y^7—3x^6—2.5x^5+10x^4—x^3—9x^2+2x=2$, to find x to about 9 or 10 places. (7364

—3	—2.5	10	1	— 9	2	2	(2.62599-
2	—2	— 9	2	6	—6	— 8	Root
—1	—4.5	1	3	—3	—4	10	
2	2	— 5	—8	—10	— 26	6.3112896	
1	—2.5	— 4	—5	—13	— 30	3.6887104	
2	6	7	6	2	40.518816	2.7866270	
3	3.5	3	+1	—11	10.518816	.9020834	
2	10	27	60	78.53136	120.543936	.7498790	
5	13.5	30	61	67.53136	131.06275|2	1522044	
2	14	55	69.8856	133.37520	8.26860	1373076	
7	27.5	85	130.8856	200.90656	139.33135	148968	
2	18	31.476	91.4064	202.79040	8.46560	137700	
9	45.5	116.476	222.2920	403.69696	147.7969|5	11268	
2	6.96	35.868	115.6920	9.73325	2.1789	10710	
11.6	52.46	152.344	337.9840	413.43021	149.9758	558	
6	7.32	40.476	142.8720	9.850	2.1915	459	
12.2	59.78	192.820	480.8560	423.280	152.167|3	99	
6	7.68	45.300	5.8064	9.968	397	92	
12.8	67.46	238.120	486.662|4	433.248	152.564	7	
6	8.04	50.340	5.843	2.529	397	6	
13.4	75.50	288.460	492.5|06	435.7|7|7	152.96|1	1	
6	8.40	1.859	5.9	2.5	4		
14.0	83.90	290.319	498.4	438.3	1|5|3.|0|0		
6	8.96	1.86	504.3	140.8			
14.6	92.66	292.17	1.5	.5			
6	30	1.9	505.|8	4|4|1.|3	296		
1|5.|2	92.9|6	294.0	5|0|7		2|9|8		
	9|3.|2	2					

23. Given $\sqrt[3]{(7x^3+4x^2)}=28—\sqrt{(20x^2—10x)}$ to find x.

Cubing $7x^3+4x^2=21952—2352\sqrt{(20x^2—10x)}+1680x^2—840x+(10x—20x^2)\sqrt{(20x^2—10x)}$. Transfer $7x^3—1676x^2+840x=21952+\sqrt{(20x^2—10x)}(10x—20x^2—2352)$. Or $7x^3—1676x^2+840x—21952=\sqrt{(20x^2—10x)}(10x—20x^2—2352)$. Again, $(7x^3—1676x^2+840x—21952)^2=(20x^2—10x)(10x—20x^2—2352)^2$. By actual involution and multiplication,

$$49x^6—23464x^5+2820736x^4—3123008x^3+74288704x^2—36879360x+481890304=8000x^6—12000x^5—1887600x^4—1882600x^3+111108480x^2—55319040x;$$

Therefore, by transposition we shall have the following equation of the 6th degree, or order: $7951x^6+11464x^5-933136x^4+$
$1240408x^3+36819776x^2-18439680x=481890304$.

24. Given $7951x^6+11464x^5-933136x^4+1240408x^3+36819776x^2-1843968x=481890304$ to find x to 9 or 10 places.

7951 Root, 4.510662488

11464	—933136	1240408	36819776	—18439680	481890304
31804	173072	—3040256	—7199392	118481536	400167425
43268	—760064	—1799848	29620384	100041856	81722880
31804	300288	—1839104	—14555808	60258304	80020123
75072	—459776	—3638952	15064576	160300160	1702757
31804	427504	—129088	—15072160	—259915	1596963
106876	— 32272	—3768040	—7584	16004024|5	105794
31804	554720	2089792	—512245	—33946	95815
138680	522448	—1678248	—51982|9	15970078	9979
31804	681936	653758	—15909	—445	9582
170484	1204384	—102449|0	—6789|1	1596963|3	397
31804	103132	78632	2210	— 43	319
202288	130751|6	—3181|7	—457|9	1|5|9|6|9|20	78
3976	10512	7600	12		64
20626|4	14126|4	441|8	—44|5		14
398	1071	813	1		13
2102|4	1519|7	125|6	—4|3		1
40	109	2			
214|2	162|9	1|2|7	{4-510662488, root as required.}		

Given $x^5-17x^4+54=350$, or $x^3-6x=6$, to find x.

—17 —17 Ans· +0 Ans.
14 54 350(14.95406861 —6 6(2847322
—3×14 —42 168 Root 2 ··· 4 —4 Root
Divisor. 12 182 Divisor.—2 10
196 170.379 4 9.152
25.9 ···· 23.31 11.621 68 ···· 5·44 848
Divisor 189.31 10.740875 Divisor . 11.44 714304
81 .880125 64 133696
27.75 ···· 1·3875 ·865276 8·44 ···· 3376 127796
Divisor 214.8175 14849 Divisor. 17.8576 5900
25 12986 16 5496
2\|7·8\|54. ···· 1114 1863 8·5\|27 ······597 404
Divisor. 216.318\|9 1731 Divisor. 18·2565 366
2 132 3 38
Divisor. 2'1\|6\|·4\|3\|1 130 Divisor. 1\|8\|3\|1\|9 37

Given $x^3+\frac{8}{9}x^2-5x=-\frac{8}{3}$ to find x.

```
                 ·8
 —5      —2·6̇(1.351029
  1·8  —3·1       8502
 —3·1     ·4̇
  1       ·3103̇
4.18 1·256  ·1341
  1·03₄   ·1311527
    9       29583̇
4.838 24194̇  28724398̇
  2·62305̇     858934
      25      575496
4.9|3|96 49398  283438
  2·8724398̇   258973
         1     24465
        10     23020
 2.|8|7|7|4|8   1445
                1438
                   7
                   6
```

Given $x^3-1.8\dot{3}x^2+.\dot{3}x=-.36$, to find x.

```
            1·8₃                 Root
       ·3̇  .    —·36(·760059217570146
1.13̇—·793   —·322
     —·46    — 446̇
      ·49   .— 446239
.326   1959     —426̇
   —·74473̇      —36025549875
        36     .—66411167916
.446716  2233583 —64843617393
   — ·7205109975  —1567550523
              25  —1440961054
.44|6|8|2   20214   —126589469
   - ·7204846377    — 72048043
              89    —54541426
  — ·720480527      —50433631
  — ·|7|2|0|4|8|0|4|3|8  —4107795
                        —3602402
                   337  —505393
                   288  — 504336
                    49    —1057
                    43     —720
```

A, B and C put in \$65, and A's money was in trade 4, B's 5, and C's 6 months. When they divided their stock and profit, A' stock and gain was \$42, B's stock and gain \$30, and C's gain \$ 9. What did each put in for his share?

By taking the ratio of each man's gain, divided by his time, and then reducing to a common denominator,

$\left\{(\frac{42-x}{4})+(\frac{30-y}{5})+\frac{9}{6}\right\}$ will give the following Rule viz.:

Give each man's gain or loss and time, and the whole stock. to find each mans particular stock.

Rule: Multiply each man's gain or loss into all the times except his own, then, as the sum of these products is to the whole stock, so is each man's product to his stock.

Let x and y denote the stock A and B put in. Then

$$\left.\begin{array}{l} 42-x=\text{A's gain}=(42-x)5.6=1260-30x \\ 30-y=\text{B's gain}=(30-y)4.6=\ 720-24y \\ 9\ \ =\text{C's gain}\quad 9.4.5\ =\ \underline{180} \end{array}\right\}\text{A, B and C's Prod.}$$

$$2160-30x-24y:65::\left\{\begin{array}{l}1260-30x:x\\ 720-24y:y\end{array}\right.$$

$$\text{Or}\left\{\begin{array}{l}2160x-30x^2-24xy=65(1260-30x)=81900-1950x\\ 2160y-30xy-24y^2=65(\ 720-24y)=46800-1460y\end{array}\right\}$$

$$\left\{\begin{array}{l}4110x-30x^2-24xy=81900\\3720y-30xy-24y^2-46800\end{array}\right\}. \quad \left\{\begin{array}{l}685x+5x^2-4xy=13650\\620y-5xy-4y^2=7800\end{array}\right\}$$

Then $685x-5x^2-13650=4xy$, and $y=\dfrac{685v-5x^2-13650}{4x}$, and by substituting this value of y in the following equation, $620y-5xy=7200+4y^2$, and reducing by ex. 6, page 15, we get $x^3-443z^2+31500x=573300$, and find $x=30=$ A's stock.

Given $3x^5-9x^4-2x^3+13x^2+2x=10$, to 8nd x, or $x^5-3x^4-.6x^3+4.\dot{3}x^2+.\dot{6}x=3.\dot{3}$.

−3
2
−1
2
1
2
3
2
5
2
7.6
6
8.2
.6
8.8
6
9.4
6
10.07
7
10.14
7
10.21
7
10.28
7
10.35

— .6
—2
—2.6
—2
— .6
6
5.3
10
15.3
4.56
19.893
4.92
24.813
5.28
30.093
5.64
35.733
7049
36.43823
7098
37.14803
7147
37.86273
7196
38.5823
310
38.613|3

4.3
−5.3
−1
1.3
−2.3
10.6
8.3
11.936
20.2693
14.888
35.1573
18·056
53.2133
2.5506763
55.7640096
2.6003623
58.364372
2.6503913
61.014763
115840
61.120603
11593
61.23653
11602
61.35255
154
61.3680
15
6|1.3|8|3

.6
−2
−1.3
−4.6
−6
12.1616
6.1616
21.0944
27.2560
3.903480675
31.159480676
4.08560604
35.244986716
18336181
35.42834852
18370960
35.6120581|2
245472
35.6366053
245532
35.661158|5
1841
35.663000
1841
35.66484|1
4
3|5.6|6|4|8|8

$$3.\dot{3}\left\}\begin{array}{r}80046\\2.6734374\end{array}\right\}$$
−2.6
6
.369696
.230304
.218116364736
.12187635263
10628504556
1559130707
1425464212
133666495
106989000
26677495
24965516
1712079
1426595
285484
285319
165
143
22
21
1

Given $x^8 + 10x^7 + 21x^6 - 55x^5 - 100x^4 + 525x^3 + 804x^2 - 630x = 216$ to find x.

<table>
<tr><th></th><th></th><th></th><th></th><th></th><th></th><th></th><th>root</th></tr>
<tr><td>10</td><td>21</td><td>—55</td><td>—100</td><td>525</td><td>804</td><td>—630</td><td>216(.79128784748</td></tr>
<tr><td>.7</td><td>7.49</td><td>19.943</td><td>— 24.5399</td><td>—87.17793</td><td>306.475449</td><td>777.3328143</td><td>103.13297001</td></tr>
<tr><td>10.7</td><td>28.49</td><td>—35.057</td><td>—124.5399</td><td>437.82207</td><td>1110.475449</td><td>147.3328143</td><td>112.86802999</td></tr>
<tr><td>.7</td><td>7.98</td><td>25.529</td><td>— 6.6696</td><td>—91.84665</td><td>242.182794</td><td>946.8607701</td><td>111.09195214</td></tr>
<tr><td>11.4</td><td>36.47</td><td>— 9.528</td><td>—131.2095</td><td>345.97542</td><td>1352.658243</td><td>1094.1935844</td><td>1.77507785</td></tr>
<tr><td>.7</td><td>8.47</td><td>31.458</td><td>15.3510</td><td>—81.10095</td><td>185.412129</td><td>140.1614394</td><td>1.37786939</td></tr>
<tr><td>12.1</td><td>44.94</td><td>21.930</td><td>—115.8585</td><td>264.87447</td><td>1538.070372</td><td>1234.355023|8</td><td>.49720836</td></tr>
<tr><td>.7</td><td>8.96</td><td>37.730</td><td>41.7620</td><td>—51.86755</td><td>19.278955</td><td>141.917499</td><td>27595723</td></tr>
<tr><td>12.8</td><td>53.90</td><td>59.660</td><td>— 74.0965</td><td>213.00692</td><td>1557.34932|7</td><td>1376.272523</td><td>12125114</td></tr>
<tr><td>7</td><td>9.45</td><td>44.345</td><td>72.8035</td><td>1.20369</td><td>19.51177</td><td>1.596962</td><td>11041868</td></tr>
<tr><td>13.5</td><td>63.35</td><td>104.005</td><td>— 1.2930</td><td>214.2016|1</td><td>1576.86110</td><td>1377.86948|5</td><td>1083245</td></tr>
<tr><td>7</td><td>9.94</td><td>51.303</td><td>14.6673</td><td>2.5869</td><td>19.8748</td><td>1.59719</td><td>966260</td></tr>
<tr><td>14.2</td><td>73.29</td><td>155.308</td><td>13.3743</td><td>216.797|5</td><td>1596.7359</td><td>1379.46667</td><td>116985</td></tr>
<tr><td>7</td><td>10.43</td><td>7.662</td><td>15.368</td><td>4.033</td><td>.2265</td><td>31949</td><td>100430</td></tr>
<tr><td>14.9</td><td>83.72</td><td>162.97|0</td><td>28.74|3</td><td>220.83|1</td><td>1596.062|4</td><td>1379.7861|6</td><td>6555</td></tr>
<tr><td>7</td><td>1.41</td><td>7.79</td><td>16.07</td><td>5.54</td><td>226</td><td>1380.1057</td><td>5522</td></tr>
<tr><td>15.6</td><td>85.1|3</td><td>170.7|6</td><td>44.8|1</td><td>226.37</td><td>1597.18|9</td><td>1278</td><td>1033</td></tr>
<tr><td>1</td><td>1.4</td><td>7.8</td><td>16.8</td><td>8</td><td>1597.41</td><td>1380.133|5</td><td>966</td></tr>
<tr><td>1|5.|7</td><td>86.|5</td><td>178.|6</td><td>66.|6</td><td>2|2|6.4|5</td><td>5</td><td>1380.361</td><td>67</td></tr>
<tr><td></td><td>2</td><td>8.</td><td>18</td><td></td><td>1|5|9|7|4|6</td><td>11</td><td>55</td></tr>
<tr><td></td><td>8|8</td><td>18|7</td><td>7|9</td><td></td><td></td><td>1380.372</td><td>12</td></tr>
<tr><td></td><td>9</td><td>20</td><td></td><td></td><td></td><td>1380.38</td><td>11</td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td></tr>
</table>

A31 Given $x^6+2x^5+3x^4+4x^3+5x^2+6x=654321$, to find x.

3	4	5	6	654321(8.9569
80	664	5344	42792	342384
83	668	5349	42798	311937
144	1816	19872	201768	288370.478241
227	2484	25221	244566	23566.521759
208	3480	47712	75845.64249	20640.905917
435	5964	72933	320411.64249	2925.615842
272	5656	11339.9361	86879.31345	2514.350990
707	11620	84272.9369	407290.95594	411.264852
336	979.929	12259.6344	5527.16239	377.850317
1043	12599.929	96532.5705	412818.1183|3	33.414535
45.81	1021.887	13217.7510	5566.9694	29.395994
1088.81	13621.816	109750.3215	418388.0877	4.018541
46.62	1064.574	792.9263	673.4017	3.779571
1135.43	14686.390	110543.2478	419058.498|4	.238970
47.43	1107.990	796.141	673.989	209977
1182.86	15794.380	111339.38|8	419732.487	28993
48.24	64.146	799.36	101.198	25197
1231.10	15858.526	112138.75	419833.68|5	3796
49.05	64.29	96.36	101.21	3780
1280.15	15922.81	112235.1|1	419934.90	16
2.77	64.4	96.4	7.87	13
1282.9|2	15987.|2	112331.5	419942.7|7	3
2.8	65	96	7.9	3
1285.7	16052	112428	419950.7	Ans.
3	8	14	1.0	8.956979569036
128|8	1606|0	11244|2	419951.7	
1|2|9	1|6|0|7	1|1|2|4|5	4|1|9|9|3	

1 | 2 8 | 10 8 | 18 8 | 26 8 | 34 8 | 42 8 | 50.9 9 | 51.8 9 | 52.7 9 | 53.6 9 | 54.5 9 | 55.4

A man and his wife could drink a barrel of beer in 15 days; but after drinking together 6 days, the woman alone drank the remainder in 30 days. In what time could either alone drink the whole barrel?

Since the man and his wife had been drinking together 6 days, and they could drink the whole in 15 days, it is evident they drank $\frac{6}{15}$ or $\frac{2}{5}$ of the whole; and the woman drank $1-\frac{2}{5}$ or $\frac{3}{5}$ in 30 days, or the whole in 50 days, or 3 times in 150 days. And by the question, they could both drink the whole 10 times in 150 days, since the man alone could drink the whole 7 times in 150 days, or once in $21\frac{3}{7}$.

32. Given $x^5 = 125000$, to find x, or to extract the sursolid of 125000.

<table>
<tr><td></td><td></td><td></td><td></td><td>Root.</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>125000(10.456395526</td></tr>
<tr><td>10</td><td>100</td><td>1000</td><td>10000</td><td>100000</td></tr>
<tr><td>10</td><td>100</td><td>1000</td><td>10000</td><td>25000</td></tr>
<tr><td>10</td><td>200</td><td>3000</td><td>40000</td><td>21665.29024</td></tr>
<tr><td>20</td><td>300</td><td>4000</td><td>50000</td><td>3334.70976</td></tr>
<tr><td>10</td><td>300</td><td>6000</td><td>4163.2256</td><td>2952.90353</td></tr>
<tr><td>30</td><td>600</td><td>10000</td><td>54163.2256</td><td>381.80623</td></tr>
<tr><td>10</td><td>400</td><td>408.064</td><td>4329.7024</td><td>358.16663</td></tr>
<tr><td>40</td><td>1000</td><td>10408.064</td><td>58492.8280</td><td>23.63960</td></tr>
<tr><td>10</td><td>20.16</td><td>416.192</td><td>565.1425</td><td>19.92992</td></tr>
<tr><td>50.4</td><td>1020.16</td><td>10824.256</td><td>59058.9705</td><td>5.70968</td></tr>
<tr><td>4</td><td>20.32</td><td>424.384</td><td>567.860</td><td>5.37937</td></tr>
<tr><td>50.8</td><td>1040.48</td><td>11248.360</td><td>59625.930</td><td>33031</td></tr>
<tr><td>4</td><td>20.48</td><td>54.210</td><td>68.509</td><td>29886</td></tr>
<tr><td>51.2</td><td>1060.96</td><td>11302.850</td><td>59694.43|9</td><td>3145</td></tr>
<tr><td>4</td><td>20.64</td><td>54.34</td><td>68.54</td><td>2989</td></tr>
<tr><td>51.6</td><td>1081.60</td><td>11356.1|9</td><td>59762.98</td><td>156</td></tr>
<tr><td>4</td><td>2.60</td><td>54.5</td><td>3.43</td><td>120</td></tr>
<tr><td>52.0</td><td>1084.2|0</td><td>11411.6</td><td>59766.4|1</td><td>36</td></tr>
<tr><td></td><td>2.6</td><td>6.5</td><td>3.4</td><td>35</td></tr>
<tr><td></td><td>1086.|8</td><td>11418.|1</td><td>59769.8</td><td>1</td></tr>
<tr><td></td><td>1|0|9|2</td><td>1|1|4|2|4</td><td>1</td><td></td></tr>
<tr><td></td><td></td><td></td><td>59770.8</td><td></td></tr>
</table>

33. Given $x^3-16\frac{1}{3}x=19\frac{11}{27}$, to find x, or $x^3-16.\dot{3}x=19.\dot{4}0\dot{7}$.

<table>
<tr><td></td><td>$-16.\dot{3}$</td><td>0.</td><td></td></tr>
<tr><td>—1....</td><td>1</td><td>$19.\dot{4}0\dot{7}$</td><td>(—1.333333</td></tr>
<tr><td></td><td>$-15.\dot{3}$</td><td>$15.\dot{3}$</td><td>or $-1.\dot{3}$</td></tr>
<tr><td></td><td>1</td><td>$4.\dot{0}7\dot{4}$</td><td></td></tr>
<tr><td>—3.3....</td><td>.99</td><td>3.703</td><td></td></tr>
<tr><td></td><td>$-12.34\dot{3}$</td><td>$.371\dot{0}7\dot{4}$</td><td></td></tr>
<tr><td></td><td>9</td><td>334363</td><td></td></tr>
<tr><td>—3.93....</td><td>1179</td><td>36711</td><td></td></tr>
<tr><td></td><td>$-11.1454\dot{3}$</td><td>33044</td><td></td></tr>
<tr><td></td><td>9</td><td>3667</td><td></td></tr>
<tr><td></td><td>0.120</td><td>3301</td><td></td></tr>
<tr><td></td><td>—11.014</td><td>366</td><td></td></tr>
<tr><td></td><td>—1|1.|0|0|2</td><td>330</td><td></td></tr>
<tr><td></td><td></td><td>36</td><td></td></tr>
<tr><td></td><td></td><td>33</td><td></td></tr>
<tr><td></td><td></td><td>3</td><td></td></tr>
<tr><td></td><td></td><td>3</td><td></td></tr>
</table>

Given $x^4 - 34{,}62x^3 + 300{,}6357x^2 - 34{,}62x = -299{,}6357$ to find one of the values x, Ans. $x =$
12

Given $x^3 - 27x = 36$, to find x.

0 root.
−17 36(5.7658216
5.....25 −10
Div. −2 46
25 41.293
15.7...10.99 4.707
Div. 58.99 4.289976
49 417024
17.16...1.0296 363096
Div. 71.4996 53928
36 50903
864 3025
Div. 72.6192 2909
12 16
Div. 72.718 73
Div. 7,2.7,3 43
43

Given $x^3 + 5x^2 + 29x = 1829$, to find x.

+5
29 1829(10.09024
15....150 1790 [296
Div. 179 39
100 38.894229
35.09...3.1581 105771
Div. 432.1581 87066
81 18705
35|2702....71 17414
Div. 435.3314 1291
4,3,5,.3,4 871
420
392
28
26

10.09024296 root true to the last.

Given $x^3 - 9x = 12$, to find x.

−9 12(3.5223339
3....9 0
Div. 0 12
9 11.375
9.5 ..4.75 625
Div. 22.75 559208
25 65792
10.52 . . 2104 56384648
Div. 27.9604 9407352
4 8464986
10.562 ..21124 942366
Div. 28.192324 846603
4 95763
3169 84661
Div. 28.216621 96 11102
32 85 8466
Div. 28.22011 11 2636
Div. 2,8.2,2,0,4 9 2540

Given $x^3 - 5x^2 + 2x = -12$, to find x.

−5
+2 −12(−1.22606
−6...+6 −8 [79
Div. +8 −4
+1 −3.328
−8.2....1.64 −672
Div. 16.64 −557367
4 −114633
−8.63....2589 −113345
Div. 18.5789 −1288
9 −1137
−8.696....522 18 −151
Div. 18.8909 17 −133
1,8,.9,4,3 18

Two farmers sell two sorts of corn: A sells 5 bushels; B rereceives in all for his $36. Now, says B to A, if we add the number of my bushels to the number of your dollars, the sum will be 28. Says A to B, and if I add the square of my dollars to the square of your bushels, the sum will be 424. How many bushels did B sell, and how many dollars did A receive? See Index.

Ans. B sold 18 bushels, and A received $10.

Given $3x - x^3 = 1$, or $x^3 - 3x = -1$, to find x true to the last place.

```
                    0        Root.
    — 3.        — 1(.347296355
3       .09     — 0.873
    — 2.91      — .127
        .09     — .107696
.94     376     — 19304
    — 2.6924    — 18522077
         16       — 781923
  1.027 ... 7189  — 527713
    — 2.646011    — 254210
             49   — 237444
  1.04|12.. 208     — 16766
  — 2.63856,5       — 15829
1|0|4169 .... 94  — 14 — 937
  2.|6|3|8|2|6|3  — 13 — 791
                       — 146
                       — 132
```

Given $x^3 + 24,84x^2 - 67.613x = 3761.2758$, to find x to about 10 places of figures.

```
                          +24.84
       — 67.613   3761.2758(11.1
34.84    348.4    2807.87          9733222
Div.     280.787   953.405
         100.      785.027
55.84 .. 55.84     168.3788
Div.     785.027    84.7661
           1        83.6127
57.94 ... 5.794     77.283513
Div.     847.661     6.329187
            1        6.050544
58.23 ... 5.2407      .278643
Div.     858.7057     .259437
             81        19206
58.41 .... 4089        17296
Div.     864.3634       1910
              17        1730
         864.789         180
        8|6|4|.8|1       173
                           7
```

Given $x^3 - 15x^2 + 63x = 50$, to find x.

```
                 — 15
         63.     50(1.02803923
— 14  — 14.      49          Root.
  Div.  49.       1.
         1.       .715208
— 11.98 — .2396   .284792
Div.    35.7604   283405952
              4     1386048
— 11,932.. — 95456  1059900
Div.    35.425744    326148
               64    317966
— 11.9|1597 — 358   56  8182
Div.    35.329994   35  7066
        3|5|3|2|9|6 21  1116
                    21  1060
```

Given $x^3 + 9x = 30$, to find the value of x.

```
                   0.
          9       30(2·1808497702
  2···· 4         26          Root. 559
        13         4·
         4         2·161
  6·1 ···· 61      1.839
Divisor 21·61      1.819232
              1      19768
6·38 ···· 5104     18609946112
Divisor 22·7404     1158053888
              64     930717101
6·5408 ····523264    227336787
Divisor 23·26243264  209414233
                 64   17922554
6.5|42|44 ···· 26170  16287818
Divisor 23.26792752    1634736
6.54|2529 ···· 5888    1628782
Divisor 23.26824810       5954
6. 5 425477 ····458       4654
Divisor 23.26831156       1300
                 46       1163
D.  2|3|.2|6|8|3|2         137
                           116
                            21
```

32. Given $x^4 - 3x^2 - 4x = 3$, to find x.

0	−3	−4	3(2.3027756377637	
2	4	2	−4	Root.
2	1	−2	7	
2	8	18	6.9141	
4	9	16	859	
2	12	7.047	61851033616	
6	21	23.047	24048966384	
2	2.49	7.821	21702262121	
8.3	23.49	30.868	2346704263	
3	2.58	57516808	2171779004	
8.6	26.07	30.925516808	174925259	
3	2.67	57553624	155137157	
8.9	28.74	30.983070432	19788102	
3	18404	20181169	18616668	
9.202	28.758404	31.003231601	1171434	
2	18408	20165682	930533	
9.204	28.776812	31.02339728|3	240901	
2	18412	201706	217102	
9.206	28.795224	31.0254143|4	23699	
2	6446	20171	21720	
9.2|0|8	28.801670	31.027431|4	1979	
	6447	172|8	1862	
	28.808117	31.02760|4	117	
	6447	17|2	93	
	28.856	31.0|42|7|7|8	24	
	64		22	
	28.8152|0		2	
	6			
	28.8158			

33. Given $x^4 + 5.5432x^3 - 38.89x^2 = 1829.4194$, to find x,

·x			Root (81.
5.5432	38.89	—98.2177	1829.4194(7.07644
7.	87.8024	342.3868	1709.1837
12.5432	48.9124	244.1691	120.2357
7.	136.8024	1300.0026	109.9240
19.5432	185.7148	1544.1717	10.3117
7.	185.8024	26.1709	9.5940
26.5432	371.5172	1570.34|26	7177
7.	2.3529	26.33|6	6406

33.5432	373.87|01	1596.6|8	771
7	2.36	2.2|7	640
33.61|32	376.23	1599.0	131
7	2.36	2.3	129
33.68	378.6	1601.3	2
7	2	2	
33.|7	378.8	1|6|0|1.|5	$x=70764481.$
	3|7|9.		

34. Given $x^4 - 17x^2 - 20x = 6$, to find x.

0	−17	−20	6(4.645751	
4	16	−4	−96	Root.
4	−1	−24	102	
4	32	124	92.0256	
8	31	100	9.9744	
4	48	53.376	8.6949	
12	79	153.376	1.2795	
4	9.96	59.568	1.1120	
16.6	88.96	212.944	1675	
.6	10.32	4.428	1561	
17.2	99.28	217.372	114	
.6	10.68	4.456	112	
17.8	109.96	221. 83	2	
.6	74	56	2	
1|8.|4	110.7|0	222. 4		
	7	6		
	111.4	2|2|3.		
	11|2.1			

35. Given $246x^4 + 4701x^3 - 9800x^2 + 4754x = -101$, to find x.

246	4701	−9800	4754	−101(·9681	Root
	221.4	4430.16	−4832.856	−70.9704	
	4922.4	−5369.84	−78.856	−30.0296	
	221.4	4629.42	666.378	−28.7864	
	5143.8	−740.42	−745.234	−1.2432	
	221.4	4828.68	265.460	−1.2235	
	5365.2	4088.26	−579.774	−197	
	221.4	336.08	285.67	−111	
	5586.6	4424.3|4	−194.10	−86	
	14.8	337.	41.16	−77	
	5601.4	4761.3	−152.9|4	−9	
	15	337.9	41.5	−9	

Of Cubic Equations.

95. A cubic equation is that in which the unknown quantity rises to 3 dimensions; and, like quadratics, or those of the higher orders, is either simple or compound. A simple or pure cubic equation is of the form $ax^3=b$, or $x^3=\frac{b}{a}$; where $x=\sqrt[3]{\frac{b}{a}}$.

A compound cubic equation is of the form

$$x^3+ax=b,\ x^3+ax^2=b,\ \text{or}\ x^3+ax^2+bx=c,$$

in each of which the known quantities a, b, c, may be either $+$ or $-$.

Or either of the two latter of these equations may be reduced to the same form as the first, by taking away its second term; which is done as follows:

Rule. Take some new unknown quantity, and subjoin to it a third part of the coefficient of the second term of the equation with its sign changed; then, if this sum, or difference, as it may happen to be, be substituted for the original unknown quantity and its powers in the proposed equation, there will arise an equation wanting its second term.

Note. The second term of any of the higher orders of equations may also be exterminated in a similar manner, by substituting for the unknown quantity some other unknown quantity, and the 4th, 5th, &c. part of the coefficient of its second term, with the sign changed, according as the equation is of the 4th, 5th, &c. power.

1. Required to exterminate the second term of the equation $x^3+3ax^2=b$, or $x^3+3ax^2-b=0$. Here $x=z-\frac{3a}{3}=z-a$.

$$\text{Then}\ \left\{\begin{array}{l} x^3=z^3-3az^2+3a^2z-a^3 \\ 3ax^2-\quad +3az^2-6a^2z+3a^2 \\ -b= \qquad\qquad\qquad -b \end{array}\right\};$$ whence $z^3-3a^2z+2a^3-b=0$; or $z^3-3a^2z=b-2a^3$, in which equation the second power (z^2) of the unknown quantity is wanting.

2. Let the equation $x^3-12x^2+3x=-16$ be transformed into another that shall want the second term. Here $x=z+4$.

$$\text{Then}\ \left\{\begin{array}{l} (z+4)^3=z^3+12z^2+48z+64 \\ -12(z+4)^2=\ -12z^2-96z-192 \\ +2(z+4)= \qquad\qquad +3z+12 \end{array}\right.$$

Whence $z^3-45z-116=-16$; or $z^3-45z=100$, which is an equation where z^2, or the second term, is wanting, as before.

3. Let the equation $x^3-6x^2=10$ be transformed into another that shall want the second term. Ans. $z^3-12z=26$.

4. Let $y^3-15y^2+81y=243$ be transformed into an equation that shall want the second term. Ans. $x^3+6x=88$.

5. Let the equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$ be transformed into another that shall want the second term. Ans. $y^3+\frac{11}{16}x=\frac{3}{4}$.

$$\begin{array}{rl} \multicolumn{2}{c}{\text{Here } y=x+5,} \\ y^3= & x^3+15x^2+\ \ 75x+125 \\ -15y^2= & \quad\ -15x^2-150x-375 \\ +81y= & \qquad\qquad\ +\ 81x+405 \\ -243= & \qquad\qquad\qquad\quad -243 \end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\end{array}}\right\}\left\{\begin{array}{rl} \multicolumn{2}{c}{\text{Here } x=y-\frac{1}{4},} \\ x^3= & y^3-\frac{3}{4}y^2+\frac{3}{16}y-\frac{1}{64} \\ +\frac{3}{4}x^2= & \quad\ +\frac{3}{4}y^2-\frac{3}{8}\,y+\frac{3}{64} \\ +\frac{7}{8}x= & \qquad\qquad\ \ \frac{7}{8}\,y+\frac{7}{32} \\ -\frac{9}{16} & \qquad\qquad\qquad -\frac{9}{16} \end{array}\right.$$

$x^3+6x-88=0$. Hence we have $y^3+\frac{11}{16}y-\frac{3}{4}=0$.

6. Let $x^4+8y^3-5x^2+10x-4=0$ be transformed into another that shall want the second term. Ans. $y^4-29y^2+94y-92=0$.

Here $x=y-2$,

$$\begin{array}{rl} x^4= & y^4-8y^3+24y^2-32y+16 \\ 8x^3= & \qquad\ 8y^3-48y^2+96y-64 \\ -5x^2= & \qquad\qquad\quad -5y^2+20x-20 \\ +10x= & \qquad\qquad\qquad\qquad\ 10y-20 \\ -4= & \qquad\qquad\qquad\qquad\qquad\ -4 \end{array}$$

We have $x^4-29y^2+94y-92=0$.

7. Let $x^4-3x^3+3x^2-5x-2=0$ be transformed into another that shall want the third term. Ans. $y^4+y^3-4y-6=0$.

Here $x=y+e$,

$$\begin{array}{rl} x^4= & y^4+4ey^3+6e^2y^2+4e^3y+e^4 \\ -3x^3= & \qquad\ -\ 3y^3-9e\,y^2-9e^2y-3e^3 \\ +3x^2= & \qquad\qquad\quad +\ 3y^2+6e\,y+3e^2 \\ -5x= & \qquad\qquad\qquad\qquad -5\ y-5e \\ -2= & \qquad\qquad\qquad\qquad\quad -2 \end{array}$$

Now, the value of e, by which the third term is taken away, is had by resolving the quadratic equation $6e^2-9e+3=0$; the roots of which are found $=1$, or $\frac{1}{2}$: hence, by substituting $y+1$ for x, in the given equation, we find $y^4+y^2-4y-6=0$, an equation wanting the third term.

8. Let $3x^3-2x+1=0$ be transformed into another whose roots are the reciprocals of the former. Ans. $y^3-2y^2+3=0$.

Here let $x=\dfrac{1}{y}$; then $3x^3-2x+1=\dfrac{3}{y^3}-\dfrac{2}{y}+1=0$, or $3-2y^2+y^3=0$; hence $y^3-2y^2+3=0$; the roots of which are the reciprocals of the former.

9. Let $x^4-\frac{1}{2}x^3-\frac{2}{3}x^2-\frac{3}{4}x+\frac{1}{18}$ be transformed into another, in which the coefficient of the highest term shall be unity, and the remaining terms integers. Ans. $y^4-3x^3+12y^2-162y+72=0$.

Here let $x=\dfrac{y}{6}$; then $\dfrac{y^4}{6^4}-\dfrac{y^3}{2.6^3}+\dfrac{y^2}{3.6^2}-\dfrac{3y}{4.6}+\dfrac{1}{18}=y^4-3y^3+12y^2-162y+72=0$, the equation required; the roots of which are 6 times those of the former

Of the solution of Cubic Equations.

96. **Rule.** Take away the second term of the equation when necessary, as directed in the preceding rule. Then, if the numeral coefficients of the given equation, or of that arising from the reduction above mentioned, be substituted for a and b in either of the following formulæ, the result will give one of the roots, as required.*

$$x^3+ax=b.$$

$$x=\left\{\begin{array}{l}\sqrt[3]{\left\{\frac{b}{2}+\sqrt{(\frac{b^2}{4}+\frac{a^3}{27})}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{(\frac{b^2}{4}+\frac{a^3}{27})}\right\}} \text{ or}\\ \sqrt[3]{\left\{\frac{b}{2}+\sqrt{(\frac{b^2}{4}+\frac{a^3}{27})}\right\}}-\dfrac{\frac{1}{3}a}{\sqrt[3]{\left\{\frac{b}{2}+\sqrt{(\frac{b^2}{4}+\frac{a^3}{27})}\right\}}}\end{array}\right.$$

Where, it is to be observed that when the coefficient a, of the second term of the above equation, is negative, $\frac{a^3}{27}$, as also $\frac{a}{3}$, in the formula, will be negative; and if the absolute term b be negative, $\frac{1}{2}b$ in the formula will be negative; but $\frac{1}{4}b^2$ will be positive.

It may likewise be remarked, that when the equation is of the form $x^3-ax=\pm b$, and $\frac{a^3}{27}$ is greater than $\frac{b^2}{4}$, or $4a^3$ greater than $27b^2$, the solution of it cannot be obtained by the above rule; as the question, in this instance, falls under what is usually called the Irreducible Case of cubic equations. See index.

1. Given $2x^3-12x^2+36x=44$, to find the value of x.

Here $x^3-6x^2+18x=22$, by dividing by 2; and, in order to exterminate the second term, put $x=z+\frac{6}{3}=z+2$:

$$\text{Then}\left|\begin{array}{rr}(z+2)^3= & z^3+6z^2+12z+8\\ -6(z+2)^2= & +6z^2-24z-24\\ 18(z+2)= & 18z+36\end{array}\right|=22,$$

---

* If, instead of the regular method of reducing a cubic equation of the general form $x^3+ax^2+bx+c=0$, to another, wanting the second term, as pointed out in the preceding article, there be put $x=\frac{1}{3}(y-a)$, we shall have, by substitution and reduction, $y^3+(9b-3a^2)y=9ab-27c-2a^3$; where, since the value of y can be determined by either of the formulæ given in this rule, the value of x will also be known, being $x=\frac{1}{3}(y-a)$. And if $b=0$, or the original equation be of the following form, $x^3+ax^2+c=0$, the reduced equation will be $x^3-3a^2y=-27c$, where the value of y being found as above, we shall have, as before, $x=\frac{1}{3}(y-a)$, which formulæ, it may be observed, are more convenient, in some cases, than those resulting from the preceding article; as the coefficients thus obtained are always integers; whereas, by the former method, they are frequently fractions.

Whence $z^3+6z+20=22$, or $z^3+6z=2$; and, consequently, by substituting 6 for *a* and 2 for *b*, in the first formula, we shall have
$x=\sqrt[3]{\{\frac{2}{2}+\sqrt{(\frac{4}{4}+\frac{216}{27})}\}}+\sqrt[3]{\{\frac{2}{2}-\sqrt{(\frac{4}{4}+\frac{216}{27})}\}}=$
$\sqrt[3]{\{1+\sqrt{(1+8)}\}}+\sqrt[3]{\{1-\sqrt{(1+8)}\}}=\sqrt[3]{(1+\sqrt[3]{9})}+$
$\sqrt[3]{(1-\sqrt{9})}=\sqrt[3]{(1+3)}+\sqrt[3]{(1-3)}=\sqrt[3]{4}-\sqrt[3]{2}$; therefore $x=z$
$+2=\sqrt[3]{4}-\sqrt[3]{2}+2=2+1.587401-1.259921=2.32748$, Ans.

2. Given $x^3-6x=12$, to find the value of x.

Here *a* being equal to -6, and *b* equal to 12, we shall have by the formula $x=\sqrt[3]{\{6+\sqrt{(36-8)}\}}-\frac{-2}{\sqrt[3]{\{6+\sqrt{(36-8)}\}}}=$
$\sqrt[3]{(6+\sqrt{28})}+\frac{2}{\sqrt[3]{(6+\sqrt{28})}}=\sqrt[3]{(6+5.2915)}+\frac{2}{\sqrt[3]{(6+5.2915)}}$
$=\sqrt[3]{(11.2915)}+\frac{2}{\sqrt[3]{(11.2915)}}=2.2435+\frac{2}{2.2435}=2.2435+$
$.8957=1392$; therefore $x=3.1392$, the answer.

3. Given $x^3-2x=-4$, to find the value of x.

Here *a* being $=-2$, and $b=-4$, we shall have, by the formula,
$x=\sqrt[3]{\{-2+\sqrt{(4-\frac{8}{27})}\}}+\sqrt[3]{\{-2-\sqrt{(4-\frac{8}{27})}\}}$, or by reduction,
$\sqrt[3]{(-2+\frac{10}{9}\sqrt{3})}-\sqrt[3]{(2+\frac{10}{9}\sqrt{3})}=\sqrt[3]{(-2+1.9245)}$
$-\sqrt[3]{(2+1.9245)}=\sqrt[3]{(-.0755}-\sqrt[3]{3.9245}=-1226-1.5773=$
-1.9999, or -2, Ans.

Note. When one of the roots of a cubic equation has been found, by the common formula as above, or in any other way, the other two roots may be determined as follows:

Let the known root be denoted by r, and put all the terms of the equation, when brought to the left hand side, $=0$; then, if the equation so formed be divided by $x\pm r$, according as r is positive or negative, there will arise a quadratic equation, the roots of which will be the other two roots of the given cubic.

4. Given $x^3-15x=4$, to find the three roots or values of x.

Here x is readily found, by a few trials, to be equal to 4, and therefore, by division, we get $x-4)x^3-15x-4(x^2+4x+1$;

Whence, according to the note above given, $x^2+4x+1=0$, or $x^2+4x=-1$; the two roots of which quadratic are $-2+\sqrt{3}$, and $-2-\sqrt{3}$; and consequently 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$, are the three roots of the proposed equation.

---

1. Given $x^3+3x^2-6x=8$, to find the root of the equation, or the value of x. See index. Ans. $x=2$.

2. Given $x^3-6x=6$, or $x^3+9x=6$, or $x^3+2x^2-23x=70$, or $x^3-17x^2+54x=350$, or $x^3-3x^2=5$, or $x^3+x^2=500$, to find the value of x in each expression. Answers, $\sqrt[3]{4}+\sqrt[3]{2}$, or $\sqrt[3]{9}-\sqrt[3]{3}$, or 5.1346, or 14.954, or 3.1038, and 7.61678, as required.

3. Given $x^3-6x=4$, to find the three roots of the equation, or the three values of x. Ans. -2, $1+\sqrt{3}$, and $1-\sqrt{3}$.

4. Given $x^3-5x^2+2x=-12$, to find the three roots of the equation, or values of x. Ans. -3, $1+\sqrt{5}$, and $1-\sqrt{5}$.

6. Find the root of the equation $x^3+x=500$, or $x^3-6x=5.6$. Ans. $x=7.89500828$, and $x=2.82526384$.

Solution of Cubic Equations by the Tables of Sines and Tangents.

97. Cubic Equations may often be more readily resolved by means of the following trigonometrical formulæ, than by the method before given; particularly when they fall under what is usually called the irreducible case, in which the common method is known to fail.

1. Given $x^3+ax=\pm b$, to find the value of x.

Put $\frac{2ar}{3b}\sqrt{\frac{a}{3}}=$ tan. z, and $\sqrt[3]{(r^2 \text{ tan. } \tfrac{1}{2}z)}=$ tan. u,

Then $x=\pm\frac{2}{r}\sqrt{\frac{a}{3}}\times$ cot. $2u$; where the upper or under sign is to be taken according as b is positive or negative.

2. Given $x^3-ax=\pm b$, to find the value of x.

This form has two cases, according as $\frac{2ar}{3b}\sqrt{\frac{a}{3}}$ is less or greater than radius r, or $4a^3$ less or greater than $27b^2$.

In the first of these cases, or when $4a^3>27b^2$,

Put $\frac{2ar}{3b}\sqrt{\frac{a}{3}}=$ sin. z, and $\sqrt[3]{(r^2 \text{ tan. } \tfrac{1}{2}z)}=$ tan. u;

Then $x=\pm\frac{2}{r}\sqrt{\frac{a}{3}}\times$ cosec. $2u$, where the upper or under sign is to be taken, according as b is positive or negative.

In the second case, or when $4a^3<27b^2$,

Put $\frac{3br}{2a}\sqrt{\frac{3}{a}}=$ cos. z, then x will have the three following values;

$$x=\pm\frac{2}{r}\sqrt{\frac{a}{3}}\times\text{cos.}\frac{z}{3}\text{; }x=\mp\frac{2}{r}\sqrt{\frac{a}{3}}\times\text{cos. }(60^\circ-\frac{z}{3}$$

$$\text{and }x=\mp\frac{2}{r}\sqrt{\frac{a}{3}}\times\text{cos. }(60^\circ+\frac{z}{3})$$

where the upper sign is to be used when b is positive, and the under sign when it is negative, as in the other cases.

1. Given $x^3+18x=6$, to find the only real value of x.

Here $a=18$, and $b=6$; whence $\frac{2ar}{3b}\sqrt{\frac{a}{3}}=2r\sqrt{6}=$ tan. z; and, consequently, by formula 1, we shall have

Log. 6	0.7781513
$\frac{1}{2}$ log. 6	0.3890756
Log. 2	0.3010300
Log. r	10.0000000
Log. tan. z	10.6901056

Therefore $z = 78^\circ\ 27'\ 47''$, and $\frac{1}{2}z = 39^\circ\ 13'\ 53\frac{1}{2}''$.

Secondly, $\sqrt[3]{(r^2 \text{ tan. } \frac{1}{2}z)} = \text{tan. } u$, whence

Log. r^2	20.0000000
Log. tan. $\frac{1}{2}z$	9.9119523
Sum	3)29.9119523
Log. tan. u	9.9706507

Therefore $u = 43^\circ\ 4'\ 55''$, and $2u = 86^\circ\ 7'\ 50''$.

Thirdly, $\frac{2}{r}\sqrt{\frac{a}{3}} \times \text{cot. } 2u = \frac{2}{r}\sqrt{6} \times \text{cot. } 2u$, whence

$\frac{1}{2}$ log. 6	0.3890756
Log. 2	0.3010300
Log. cot. $2u$	8.8301834
Sum	9.5202890
Log. r	10.0000000
Log. x	1.5202890

Consequently $x = .3313139$, the positive root of the equation, as required.

2. Given $x^3 - 3x = 1$, which is an equation falling under the irreducible case, to find its three roots.

Here, a being $= 3$, and $b = 1$, we shall have, by taking the radius $r = 1$, $\text{cos. } z = \frac{3b}{2a}\sqrt{\frac{3}{a}} = \frac{1}{2} = .5 = \text{cos. } 60^\circ$, whence

$$x = 2\sqrt{\frac{a}{3}} \times \text{ cos. } \frac{z}{3} = 2 \text{ cos. } 20^\circ = +1.8793852,$$

$$x = -2\sqrt{\frac{a}{3}} \times \text{ cos. } (60^\circ - \frac{z}{3}) = -2 \text{ cos. } 40^\circ = -1.5320888,$$

$$x = -2\sqrt{\frac{a}{3}} \times \text{ cos. } (60^\circ + \frac{z}{3}) = -2 \text{ cos. } 80^\circ = -.3472964.$$

Therefore the three roots are 1.8793852, —1.5320888, and —.3472964.

And if the equation be $x^3 - 8x = -1$, the three roots or values of x are —1.8793852, 1.5320888, and .3472964.

Which are the negatives of the roots of the former equation.

1. Given $x^3 - 2x = -2$, to find the root of the equation, or the value of x. Ans. $x = -1.7693$.

2. Given $x^3 - 300x = -1000$, to find the root of the equation, or the value of x. See index. Ans. $x = 3.472964$.

3. Given $x^3 - 9x = 9$, to find all the three roots of the equation, or values of x. Ans. 3.411474, —2.226682, and —1.184792.

4. Given $x^3 - x^2 - 2x + 1 = 0$, to find all the three roots of the equation, or the three values of x. Ans. $2 \text{ cos. } \frac{1}{7}n$, $-2 \text{ cos. } \frac{2}{7}n$, and $2 \text{ cos. } \frac{3}{7}n$, where $n = 180^\circ$, and rad. $= 1$.

98. **This method of cubic equations by converging series, which in some cases will be found more convenient in practice than either of the former, consists in substituting the numeral parts of the given equations in the place of the literal one, in one of the following general formulæ to which it belongs; and then collecting as many terms of the series as are sufficient for determining the value of the unknown quantity to the degree of exactness required.** 1. Given $x^3+ax=b$.

$$1.,\ x=\frac{2b}{\sqrt{}\{2(27b^2+4a^3)}\{1+\frac{2.5}{6.9}(\frac{27b^2}{27b^2+4a^3})+\frac{2.5.8.11}{6.9.12.15}(\frac{27b^3}{27b^2+4a^3})^2+\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{27b^3}{27b^2+4a^3})^3+\&c.\}$$

$$\text{or } x=\frac{2b}{\sqrt{}\{2(27b^2+4a^3)\}}\{1+\frac{2.5}{6.9}(\frac{27b^2}{27b^2+4x^3})A+\frac{8.11}{12.15}(\frac{27b^2}{27b^2+4a^3})B+\frac{14.17}{18.21}(\frac{27b^2}{27b^2+4a^3}C+\frac{20.23}{24.27}(\frac{27b^3}{27b^3+4a^3})D$$

In which case, as well as in the following ones, A, B, C, &c. denote the terms immediately preceding those in which they are first found. 2.Given $x^3-ax=\pm b$, where $\frac{1}{4}b^2$ is supposed to be greater than $\frac{1}{27}a^3$ or $27b^2-4a^3$.

$$2.\ x=\pm 2\sqrt[3]{\frac{b}{2}}\{1-\frac{2}{3.6}(\frac{27b^2-4a^3}{27b^2})-\frac{2.5.8}{3.6.9.12}(\frac{27b^2-4a^3}{27b^2})^2-\frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{27b^2-4a^3}{27b^2})^3-\&c.\},\text{ or}$$

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\{(-\frac{2}{3.6}(\frac{27b^2-4a^3}{27b^2})A-\frac{5.8}{9.12}(\frac{27b^2-4a^3}{27b^2})B-\frac{11.14}{15.18}(\frac{27b^2-4a^3}{27b^2})C-\frac{17.20}{21.24}(\frac{27b^2-4a^3}{27b^2})D-\&c.$$

In which case the upper sign must be taken when b is positive, and the under sign when it is negative; and the same for the first root in the two following cases.

3. Given $x^3-ax=\pm b$, where $\frac{1}{4}b^2$ is supposed to be less than $\frac{1}{27}a^3$, or $27b^2\times 4a^3$.

$$3.\ x=\pm 2\sqrt[3]{\frac{b}{2}}\{1+\frac{2}{3.6}(\frac{4a^3-27b^2}{27b^3})-\frac{2.5.8}{3.6.9.12}(\frac{4a^3-27b^2}{27b^3})^2+\frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{4a^3-27b^2}{27b^2})^3-\&c.\},\text{ or}$$

$$x=\pm 2\sqrt[3]{\frac{b}{2}}\{1+\frac{2}{3.6}(\frac{4a^3-27b^3}{27b^3})A-\frac{5.8}{9.12}(\frac{4a^3-27b^2}{27b^3})B+\frac{11.14}{15.18}(\frac{4a^3-27b^2}{27b^3})C-\frac{17.20}{21.24}(\frac{4a^3-27b^3}{27b^3})D+\&c.$$

Which series answers to the irreducible case, and must be used when $4a^3$ is less than $27b^2$. And if the root thus found be put $=r$, the other two roots may be expressed as follows:

$$x=\mp\frac{r}{2}\pm\frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\{1-\frac{2.5}{6.9}(\frac{4a^3-27b^2}{27b^2})+\frac{2.5.8.11}{6.9.12.15}(\frac{4a^3-27b^2}{27b^2})^2-\frac{2.5.8.11.14.17}{3.6.9.12.15.18.21}(\frac{4a^3-27b^2}{27b^2})^3+\&c,\}$$

$$x=\mp\frac{r}{2}\pm\frac{\sqrt{(4a^3-27b^2)}}{9\sqrt[3]{2b^2}}\{1-\frac{2.5}{6.9}(\frac{4a^3-27b^2}{27b^2})A+\frac{8.11}{12.15}(\frac{4a^3-27b^2}{27b^2})B-\frac{14.17}{18.22}(\frac{4a^3-27b^2}{27b^2})C+\frac{20.23}{25.27}(\frac{4a^3-27b^2}{27b^2})^2D$$

Where $-\frac{1}{2}r$, or $+\frac{1}{2}r$, must be taken according as b is positive or negative; and the double signs $\pm$ must be considered as + in one case and — in the other, as usual. 4. Given $x^3-ax=\pm b$, where $\frac{1}{4}b^2$ is still
13 supposed to be less than $\frac{1}{27}a^3$, or $27b^2=4a^3$.

$$4.\ x=\pm2\frac{2b}{\sqrt[3]{\{2(4a^3-27b^2)\}}}\{1-\frac{2.5}{6.9}(\frac{27b^2}{4a^3-27b^2})+\frac{2.5.8.11}{6.9.12.15}(\frac{27b^2}{4a^2-27b^2})^2-\frac{2.5.8.11.14.17}{6.9.12.15.18.21}(\frac{27b^2}{4a^3-27b^2})^3+\&c.$$

or}

$$x=\mp\frac{27b^2}{\sqrt[3]{\{2(4a^3-27b^2)\}}}\{1-\frac{2.5}{6.9}(\frac{27b^2}{4a^3-27b^2})A+\frac{8.11}{12.15}(\frac{.27b^2}{4a^3-27b^2})B-\frac{14.17}{18.21}(\frac{27b^2}{4a^3-27b^2})C+\frac{20.23}{24.27}(\frac{27b^2}{4a^3-27b^2})$$

D—&c., which series also answers to the irreducible case, and must be used when $4a^2$ is greater than $27b^2$. And if the root thus found be put $=r$, as before, the other two roots may be expressed thus:

$$x=\mp\frac{r}{2}\pm\sqrt[6]{\frac{4a^3-27b^2}{4}}\{1+\frac{2}{3.6}(\frac{27b^2}{4a^3-27b^2})-\frac{2.5.8}{3.6.9.12}(\frac{27b^2}{4a^3-27b^2})^2+\frac{2.5.8.11.14}{3.6.9.12.15.18}(\frac{27b^2}{4a^3-27b^2})^3-\&c.\ \text{or}$$

$$x=\mp\frac{r}{2}\pm\sqrt[6]{\frac{4a^3-27b^2}{4}}\{1+\frac{2}{3.6}(\frac{27b^2}{4a^3-27b^2})A-\frac{5.8}{9.12}(\frac{27b^2}{4a^3-27b^2})B+\frac{11.14}{15.18}(\frac{27b^2}{4a^3-27b^2})C-\frac{17.20}{21.24}(\frac{27b^2}{4a^3-27b^2})D+$$

Where the signs are to be taken as in the latter part of the preceding case.

19

Of Biquadratic Equations.

99. A biquadratic equation, as before observed, is one that rises to the fourth power, or that is of the general form $x^4+ax^3+bx^2+ax+d=0$, the root of which may be determined by means of the following formula, substituting the numbers of the given equation, with their proper signs, in the places of the literal coefficients a, b, c, d. Rule I. Find the value of z in the cubic equation $z^3+(\frac{1}{4}ac-\frac{1}{12}b^2-d)z=\frac{1}{108}b^3+\frac{1}{8}(c^2+da^2)-\frac{1}{24}b(ac+8d)$ by one of the former rules; and let the root, thus determined, be denoted by r

Then find the two values of x in each of the following quadratic equations,

$$x^2+(\tfrac{1}{2}a+\sqrt{\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\}})x=-(r+\tfrac{1}{6}b)+\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}}$$
$$x^2+(\tfrac{1}{2}a-\sqrt{\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\}})x=-(r+\tfrac{1}{6}b)-\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}},$$

and they will be the four roots of the biquadratic equation required. Or, if the equation be of the more commodious form, $x^4+bx^2+cx+d=0$, to which it can always be reduced, by taking away its second term, it may be resolved thus:

RULE II. Find the value of z in the cubic equation $z^3-(\tfrac{1}{12}b^2+d)z=\tfrac{1}{108}b^3+\tfrac{1}{8}c^2-\tfrac{1}{3}bd$, and let the root thus determined be denoted by r. Then find the two values of x, in each of the two following quadratic equations,

$$\left.\begin{array}{l} x^2+\sqrt{\{2(r-\tfrac{1}{3}b)\}}x=-(r+\tfrac{1}{6}b)+\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}} \\ x^2-\sqrt{\{2(r-\tfrac{1}{3}b)\}}x=-(r+\tfrac{1}{6}b)-\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}} \end{array}\right\} \text{ and they}$$

will be the four roots of the biquadratic equation required.

1. Given the equation $x^4-10x^3+35x^2-50x+24=0$, to find its roots. Here $a=-10$, $b=35$, $c=-50$, and $d=24$; whence, by substituting these numbers in the cubic equation

$$z^3+(\tfrac{1}{4}ac-\tfrac{1}{12}b^2-d)z=\tfrac{1}{108}b^3+\tfrac{1}{8}(c^2+da^2)-\tfrac{b}{24}(ac+8d),$$

we shall have the following reduced equation, $z^3-\tfrac{13}{12}z=\tfrac{35}{108}$, which being resolved, according to the rule before laid down for that purpose, gives $z=\tfrac{1}{6}\{\sqrt[3]{(35+18\sqrt{-3})}+\sqrt[3]{(35-18\sqrt{-3})}\}$.

But, by the rule for binomial surds, given in Art. I, Case 2, in the former part of the work, $\sqrt[3]{(35+18\sqrt{-3})}=\tfrac{7}{2}+\tfrac{1}{2}\sqrt{-3}$, and $\sqrt[3]{(35-18\sqrt{-3})}=\tfrac{7}{2}-\tfrac{1}{2}\sqrt{-3}$;

$$\text{wherefore } z=\tfrac{1}{6}\{\tfrac{7}{2}+\tfrac{1}{2}\sqrt{-3}+\tfrac{7}{2}-\tfrac{1}{2}\sqrt{-3}\}=\tfrac{7}{6}.$$

And if this number be substituted for r, -10 for a, 35 for b, and 24 for d, in the two quadratic equations,

$$x^2+[\tfrac{1}{2}a+\sqrt{\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\}}]x=-(r+\tfrac{1}{6}b)+\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}},$$
$$x^2+[\tfrac{1}{2}a-\sqrt{\{\tfrac{1}{4}a^2+2(r-\tfrac{1}{3}b)\}}]x=-(r+\tfrac{1}{6}b)-\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}},$$

they will become, after reducing them to their most simple terms, $x^3-3x=-2$, and $x^2-7x=-12$; from the first of which $x=\tfrac{3}{2}\pm\sqrt{\tfrac{1}{4}}=\tfrac{3}{2}\pm\tfrac{1}{2}=2$ or 1, and from the second $x=\tfrac{7}{2}\pm\sqrt{\tfrac{1}{4}}=\tfrac{7}{2}\pm\tfrac{1}{2}=4$ or 3; whence the four roots of the given equation are 1, 2, 3, and 4.

2. Given $x^4+12x-17=0$, to find the four roots of the equation. Here $a=0$, $b=0$, $c=12$, and $d=-17$;

Whence, by substituting these numbers in the equation $z^3-(\tfrac{1}{12}b^2+d)z=\tfrac{1}{108}b^3+\tfrac{1}{8}c^2-\tfrac{1}{3}bd$, we shall have, after simplifying the result, $z^2+17z=18$, where it is evident, by inspection, that $z=1$.

And if this number be substituted for r, 0 for b, and -17 for d, in the two quadratic equations

$$x^2+\sqrt{\{2(r-\tfrac{1}{3}b)\}}x=-(r+\tfrac{1}{6}b)+\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}},$$
$$x^2-\sqrt{\{2(r-\tfrac{1}{3}b)\}}x=-(r+\tfrac{1}{6}b)-\sqrt{\{(r+\tfrac{1}{6}b)^2-d\}},$$

they will become, after reducing them, in the usual manner, to their most simple forms,

$x^2+2^{\frac{1}{2}}x=-1+3\sqrt{2}$, and $x^2-2^{\frac{1}{2}}x=-1-3\sqrt{2}$, which being resolved, according to the general rule, we shall have,

$$x=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+3\sqrt{2})},$$
$$x=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+3\sqrt{2})},$$
$$x=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-3\sqrt{2})},$$
$$x=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-3\sqrt{2})},$$

which are the four roots of the proposed equation. Where it is to be observed that the first two are real, and the two latter imaginary.

RULE III. The roots of any biquadratic equation of the form $x^4+ax^2+bx+c=0$, may also be determined by the following general formulæ first given by *Euler*, which are remarkable for their elegance and simplicity.

Find the three roots of the cubic equation $z^3+2az^2+(a^2-4c)z=b^2$, by one of the former rules, before given for this purpose; and let them be denoted by r', r'', and r'''. Then we shall have

When b is positive,	When b is negative,
$x=\frac{-\sqrt{r'}-\sqrt{r''}-\sqrt{r'''}}{2}$	$x=\frac{+\sqrt{r'}+\sqrt{r''}+\sqrt{r'''}}{2}$
$x=\frac{-\sqrt{r'}+\sqrt{r''}+\sqrt{r'''}}{2}$	$x=\frac{+\sqrt{r'}-\sqrt{r''}-\sqrt{r'''}}{2}$
$x=\frac{+\sqrt{r'}-\sqrt{r''}+\sqrt{r'''}}{2}$	$x=\frac{-\sqrt{r'}+\sqrt{r''}-\sqrt{r'''}}{2}$
$x=\frac{+\sqrt{r'}+\sqrt{r''}-\sqrt{r'''}}{2}$	$x=\frac{-\sqrt{r'}-\sqrt{r''}+\sqrt{r'''}}{2}$

101. If the three roots r', r'', r''', of the auxiliary cubic equation be all real and positive, the four roots of the proposed equation will, also, be real; and if one of these roots be positive, and the other two imaginary, or both of them negative, and equal to each other, two of the roots of the given equation will be real, and two imaginary; which are the only cases that produce real results.

3. Given $x^4-25x^2+60x-36=0$, to find the four roots of the equation. Here $a=-25$, $b=60$, and $c=-36$; whence, by substituting these values for their equals, in the cubic equation above given, we shall have $z^3-2\times25z^2+(25^2+4\times36)z=60^2$, or $z-50z^2+769z=3600$; the three roots of which last equation, as found by trial, or by one of the former rules, are 9, 16, and 25, respectively; whence

$$x=\tfrac{1}{2}(-\sqrt{9}-\sqrt{16}-\sqrt{25})=\tfrac{1}{2}(-3-4-5)=-6,$$

$$x=\tfrac{1}{2}(-\sqrt{9}+\sqrt{16}+\sqrt{25})=\tfrac{1}{2}(-3+4+5)=+3,,$$
$$x=\tfrac{1}{2}(+\sqrt{9}-\sqrt{16}+\sqrt{25})=\tfrac{1}{2}(+3-4+5)=+2,$$
$$x=\tfrac{1}{2}(+\sqrt{9}+\sqrt{16}-\sqrt{25})=\tfrac{1}{2}(+3+4-5)=+1,$$

And ∴, the four roots of the equation are 1, 2, 3, and — 6.

1. Given $x^4-55x^2-30x+504=0$, to find the four roots or values of x. Ans. 3, 7, —4, and —6·

2. Given $x^4+2x^3-7x^2-8x=-12$, to find the four roots or values of x. Ans. 1, 2, —3, and —2.

3. Given $x^4-8x^3+14x^2+4x=8$, $x^4-17x^2-20x-6=0$, $x^4-3x-4x=3$, $x^4-{*}19x^3+123x^2-302x+200=0$, to find the four roots or values of x.

$$\text{Ans.}\left\{\begin{array}{ll}3+\sqrt{5}, & 3-\sqrt{5},\\ 1+\sqrt{3}, & 1-\sqrt{5}.\end{array}\right\}\quad\left\{\text{Ans.}\left\{\begin{array}{ll}2+\sqrt{7}, & -2-\sqrt{7}.\\ -2+\sqrt{2}, & -2-\sqrt{2}.\end{array}\right.\right.$$

$$\text{Ans.}\left\{\begin{array}{ll}\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}, & \tfrac{1}{2}-\tfrac{1}{2}\sqrt{3},\\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}, & -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}.\end{array}\right\}\quad\left\{\text{Ans.}\left\{\begin{array}{ll}1.02804, & 4{\cdot}00000\\ 6.57653, & 7.39543\end{array}\right.\right.$$

4. Given $x^4-27x^3+162x^2+356x-1200=0$, $x^4-12x^2+12x-3=0$, $x^4-36x^2+72x-36=0$, to find the four roots or values of x in each.

$$\text{Ans.}\left\{\begin{array}{ll}2.05608, & -3.0000,\\ 13.15306, & 14.7908.\end{array}\right\}\quad\left\{\begin{array}{ll}.60601, & -3.907572\\ 2.85808, & .443274\end{array}\right.$$

Ans. 0.8729836, 1.2679494, 4.7320506, —6.8729836

5. Given $x^4-5x^3+13x^2-17x+12=0$, to find the 4 roots of the equation, which are all imaginary

I have $x^2-2x+3=0$, or $x^2-3x+4=0$, Ans.

*To this we may farther add, that when one of the roots of an equation has been found, either by this method or the former, the rest may be determined as follows :

Bring all the terms to the left hand side of the equation, and divide the whole expression so formed by the difference between the unknown quantity (x) and the root first found ; and the resulting equation will then be depressed a degree lower than the given one. Find a root of this new equation, by approximation, as in the first instance, and the number so obtained will be a second root of the original equation. Then, by means of this root, and the unknown quantity, depress the equation a degree lower, and thence find a third root ; and so on, till the equation is reduced to a quadratic ; when the two roots of this, together with the former, will be the roots of the equation required.

Thus, in the equation $x^3-15z^2+63x=50$, the first root is found by approximation to be 1.02804. Hence

$$x-1.02804)x^3-15x^2+63x-50(x^2-13.97196x+48.63627=0.$$

And the two roots of the quadratic equation, $x^2-13.97196x=-48\ 63627$, found in the usual way, are 6.57653 and 7.39543.

So that the three roots of the given cubic equation $x^3-15x^2+63x=50$, are 1.02804, .6.57653, and 7.39543 ; their sum being $=15$, the coefficient of the second term of the equation, as it ought to be when they are right.

Of the resolution of Equations by approximation, or the method of successive substitutions.

102. Equations of the fifth power, and those of higher dimensions, cannot be resolved by any general rule, or algebraic formula, that has yet been discovered; except in some particular cases, where certain relations subsist between the coefficients of their several terms; or when the roots are rational, and, for that reason, can be easily found by means of a few trials. See p. 113.

In these cases, therefore, recourse must be had to some of the usual methods of approximation, among which that commonly employed is the following, which is universally applicable to all kinds of numeral equations, whatever may be the number of their dimensions; and which, by continuing the process to a sufficient length, will give the value of the root sought to any required degree of exactness. See Index.

Rule I. Find, by trials, a number nearly equal to the root sought, which call r; and let z be made to denote the difference between this assumed root and the true root x.

Then, instead of x, in the given equation, substitute its equal $r+z$, or $r-z$, according as r is too little or too great, and there will arise a new equation, involving only z and known quantities.

Reject all the terms of this equation in which z is of two or more dimensions, and the approximate value of z may then be determined by means of a simple equation.

And if the value thus found be added to r, when it is too little, or subtracted from it when it is too great, it will give a near value of x, or of the root required.

But as this approximation will seldom be sufficiently exact, the operation must be repeated, by substituting the number last found, for r, in the abridged equation exhibiting the value of z; when a second correction of z will be obtained, which being added to, or subtracted from that number, will give a nearer value of x than the former.

And by again substituting this last number for r, in the above mentioned equation, and repeating the same process as often as may be thought necessary, a value of z, and consequently of the root sought, may be found to any degree of accuracy.

Note. The decimal part of the root, as found both by this and the next rule, will in general about double itself at each operation; and therefore it would be useless, as well as troublesome, to use a much greater number of figures than these, in the several substitutions for the values of r.

1. Given $x^3+x^2+x=90$, to find the value of x, by approximation.

Here the root is soon found, by a few trials, to lie between 4 and 5; but nearer to 4 than to 5.

Let therefore $4=r$, and $x=r+z$,

$$\text{Then} \left|\begin{array}{l} x^3=r^3+3r^2z+3rz^2+z^3 \\ x^2=r^2+2rz+z^2 \\ x=r+z \end{array}\right| =90.$$

And, by rejecting the terms z^3, $3rz^2$ and z^2, or, to avoid trouble, omitting them in the operation, as being small in comparison with z, we shall have $r^3+r^2+r+3r^2z+2rz+z=90$;

Whence $z=\dfrac{90-r^3-r^2-r}{3r^2+2r+1}=\dfrac{90-64-16-4}{48+8+1}=\dfrac{6}{57}=.10$, and, consequently, $x=4.1$; nearly. Again, if 4.1 be substituted in the place of r, in the last equation, we shall have.

$$z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283,$$

And consequently $x=4.1+.00283=4.10283$, for a second approximation; and if the first four figures, 4.102, of this number, be again substituted for r, in the same equation, an approximate value of x may be obtained, to six or seven places of decimals.

And by proceeding in the same manner, the root may be found still more correctly.

1. Given $x^2+20x=100$, to find the value of x by approximation. Ans. $x=4.1421356237$.

2. Given $x^3-9x=10$, to find the value of x by approximation. Ans. 3.4494897428.

3. Given $x^3+9x^2+4x=80$, to find the value of x by approximation. Ans. $x=2.47213596$.

Here, by a few trials, the root is found to be between 2 and 3, and very nearly equal to 2.5; let, therefore, $x=r+z=2.5+z$; then

$$\begin{array}{rcl} x^3 &=& r^3+3r^2z+\&c. \\ 9x^2 &=& 9r^2+18rz+\&c. \\ 4x &=& 4r+4z \\ \hline && r^3+9r^2+4r+z(3r^2+18r+4r) \end{array}$$

$=80$, and $z=\dfrac{80-r^3-9r^2-4r}{3r^2+18r+4r}=\dfrac{-1.875}{67.75}=-.0276$. Whence $x=2.5-.0276=2.472$, the root nearly. And if this value be substituted in the above expression for z, we get $x=2.4721359$.

4. Given $x^4-38x^3+210x^2+538x+289=0$, to find the value of x by approximation. Ans. $x=30.5356537529$.

Here x is found, by trial, to be very nearly $=30$; let, therefore, $x=r+z=30+z$; then

$$\begin{aligned} x^4 &= r^4 + 4r^3z + \&c. \\ -38x^3 &= -38r^3 - 114r^2z - \&c. \\ +210x^2 &= 210r^2 + 420rz + \&c. \\ +538x &= 538r + 538z \end{aligned}$$

$r^4 - 38r^3+210r^2+538r+z(4r^3-114r^2 + 420r + 538)+289=0$, and

$$z=\frac{r^4-38r^3+210r^2+538r+289}{114r^2-4r^3-420r-538}=\frac{-10571}{-18538}=.56\,;$$

whence $x=30.56$, nearly. Again, taking 30.5 for the root, and substituting it instead of r in the expression for z, we get $z=30.5356$.

5. Given $x^5+6x^4-10x^3-112x^2-207x-110=0$, to find the value of x by approximation. Ans. 4.46410161.

By a few trials, x is found to be between 4 and 5, and nearly equal to 4.5. Let, therefore, $x=r+z=4.5+z$; then

$$\begin{aligned} x^5 &= r^5 + 5r^4z + \&c. \\ 6x^4 &= 6r^4 + 24r^3z + \&c. \\ -10x^3 &= -10r^3 - 30r^2z + \&c. \\ -112x^2 &= -112r^2 - 224rz + \&c. \\ -207x &= -207r - 207z \end{aligned}$$

$r^5+6r^4-10r^3-112r^2-207r+z(5r^4+24r^3-30r^2-227r-207)=110$; therefore

$$z=\frac{110-r^5-6r^4+10r^3+112r^2+207r}{5r^4+24r^3-30r^2-227r-207}=-.0354\,;$$

whence $x=4.5-.035=4.465$; and, taking this for the root, and substituting again in the value for z, we get $x=4.464101$, the root nearly.

103. The roots of equations, of all orders, can also be determined, to any degree of exactness, by means of the following easy rule of double position; which, though it has not been generally employed for this purpose, will be found, in certain cases, superior to the former, as it can be applied at once to any unreduced equation, consisting of surds, or compound quanties, as readily as if it had been brought to its usual form.

RULE II. Find, by trial, two numbers nearly equal to the root sought, and substitute them in the given equation instead of the unknown quantity, noting the results that are obtained from each. Then, as the difference of these results is to the difference of the two assumed numbers,

So is the difference between the true result, given by the question, and either of the former, to the correction of the assumed number belonging to the result used,

Which correction being added to that number when it is too little, or subtracted from it when it is too great, will give the root required, nearly.

And if the number thus determined, and the nearest of the two former, or any other that appears to be more accurate, be now taken as the assumed roots, and the operation be repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner, as far as may be judged necessary.

1 Given $x^3+3x^2+3x=130$, to find a near approximate value of x. Ans. $x=4.078753$.

2. Given $x^3+10x^2+5x=2600$, to find a near approximate value of x. Ans. $x=11.00673$.

It is soon found by trial that x is rather more than 11; let, therefore, 11 and 11.1 be taken for the assumed number; then

$$\begin{array}{rcr} \text{1st Sup.} & & \text{2d Sup.}\\ 1331 & & 1367.631\\ 1210 & & 1232.10\\ 55 & & 55.5\\ \hline 2596 & \text{Results} & 2655.231 \end{array}$$

$$\therefore \left\{\begin{array}{lcr} 2655.231 & 11.1 & 2600\\ 2596. & 11 & 2596\\ \hline 59.231 : & .1 :: & 4 \end{array}\right.$$

$$: \frac{4\times .1}{59.231} = .0067; \text{ whence}$$

$x=11+.0067=11.0067$, Ans.

3. Given $2x^4-16x^3+40x^2-30x+1=0$, to find a near value of x.

Dividing by 2, we have $x^4-8x^3+20x-15x=-.5$. By a few trials it is found that x is a little greater than 1; let, therefore, 1.2 and 1.3 be assumed for the numbers; then

$$\begin{array}{rr} \text{1st Sup.} & \text{2d Sup.}\\ 2.0736 & 2.8561\\ 28.8 & 33.8\\ \hline 30.8736 & 36.6561\\ -31.824 & -37.076\\ \hline -.9504\ \text{results} & -.4199 \end{array}$$

$$\therefore \left\{\begin{array}{lcr} .9504 & 1.3 & .5\\ .4199 & 1.2 & .4199\\ \hline .5305 : & .1 :: & .0801 \end{array}\right.$$

$$: \frac{.1\times .0801}{.5305} = .01509, \text{ whence}$$

$x=1.3-.0151=1.2849$. Now take 1.284 and 1.285 for the assumed numbers, and repeat the operation. Then x will come out $=1.284724$.

4. Given $x^5+2x^4-3x^3+4x^2+5x=54321$, to find a near value of x.

By a few trials x is found to be nearly $=8\frac{1}{2}$; take, therefore, 8.4 and 8.5 for the assumed numbers; then

$$\begin{array}{lrcccrl} \text{1st Sup.} & 41821.19424 & = & x^5 & = & 44370.53125 & \text{2d Sup.}\\ & 9957.4272 & = & 2x^4 & = & 10440.1250 & \\ & 1778.112 & = & 3x^3 & = & 1842.375 & \\ & 282.24 & = & 4x^2 & = & 289. & \\ & 42. & = & 5x & = & 42.5 & \\ \hline & 53880.97344 & & \text{results} & & 56984.53125 & \end{array}$$

$$\text{Therefore} \left\{\begin{array}{lcr} 56984.53125 & 8.5 & 54321\\ 53880.97344 & 8.4 & 53880.97344\\ \hline 3103.55781 : & .1 :: & 440.02656 : \end{array}\right.$$

$\therefore \dfrac{.1\times 440.02656}{3103.55781}=.0141$; whence $x=8.4+.0131=3.4141$, nearly. Taking 8.414 and 8.415 for the numbers, we have

$$\begin{array}{rcccr}
42170.867905346409824 & = & x^5 & = & 42195.935727527009375 \\
10023.976207593632 & = & 2x^4 & = & 10028.74241890125 \\
1787.017385832 & = & 4x^3 & = & 1787.654620125 \\
283.181584 & = & 4x^2 & = & 283.2489 \\
42.07 & = & 5x & = & 42.075 \\
\hline
54307.113082772041824 & & & & 54337.654666553259375 \\
54337.65465655325 & & 8.415 & & 54337.65466655325 \\
53307.11308277204 & & 8.414 & & 54321 \\
\hline
30.54158378121 & : & .001 & :: & 16.65466655325 : 000545
\end{array}$$

and $x=r-z=8.415-.000545=8.414455$, very near.

5. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$, to find an approximate value of x. Ans. 4.510661.

By trial x is found to be between 4 and 5. Let these two numbe taken for the first value; then

$$\begin{array}{lrcccr}
\text{1st Sup.} & 8\ . & = & \sqrt[3]{(7x^3+4x^2)} & = & 9.916 \quad \text{2d Sup.} \\
 & 16.73 & = & \sqrt{(20x^2-10x)} & = & 21.213 \\
\hline
 & 24.73 & & \text{results} & & 31.129
\end{array}$$

$$\text{Therefore}\left|\begin{array}{ccccccc}
31.129 & & 5 & & 28 & & \\
24.73 & & 4 & & 24.73 & & \\
\hline
6.399 & : & 1 & :: & 3.27 & : & \dfrac{3.27}{6.399}=.51;\ \text{whence}
\end{array}\right.$$

$x=4.51$, nearly. Assuming $x=4.51$ and 4.52, and repeating the operation, we get $x=4.5106$, the root, nearly.

6. Given $\sqrt{\{144x^2-(x^2-20)^2\}}+\sqrt{\{196x^2-(x^2+24)^2\}}=114$, to find a near value of x.

Here the root is found, by a few trials, to be rather more than 7; let, therefore, 7 and 7.1 be assumed for the numbers; then

$$\begin{array}{lrcccr}
\text{1st Sup.} & 7056 & = & 144x^2 & = & 7259.04 \quad \text{2d Sup.} \\
 & 4761 & = & (x^2+20)^2 & = & 4957.5681 \\
\hline
\surd & 2295 & = & 47.906 & & \surd 2302.4719=47.977 \\
\hline
 & 9604 & = & 196x^2 & = & 9880.36 \\
 & 5329 & = & (x^2+24)^2 & = & 5536.8481 \\
\hline
 & \surd(4275) & = & 65.383 & & \surd 4343.5119=65.905
\end{array}$$

$$\left.\begin{array}{rcr}
47.906 & & 47.977 \\
65.383 & & 65.905 \\
\hline
113.289 & \text{results} & 113.882
\end{array}\right\}\ \text{or}\ \left\{\begin{array}{ccccccc}
113.882 & & 7.1 & & 114. & & \\
113.289 & & 7 & & 113.882 & & \\
\hline
.593 & : & .1 & :: & .118 & : &
\end{array}\right.$$

$\dfrac{.1\times .118}{.593}=\dfrac{.0118}{.593}=.2$; whence $x=7.12$, nearly. By assuming

7.12 and 7.13 for the numbers, we get $x = 7.12209461$, the answer, nearly.

1. Given $x^3+x^2+x=100$, to find an approximate value of x.

Here it will be found, by a few trials, that the value of x lies between 4 and 5.

Hence, by taking these as the two assumed numbers, the operation will stand thus:

First Sup.		*Second Sup.*
4	x	5
16	x^2	25
64	x^3	125
84	Results	155

155....5....100
∴ 84....4.... 84
71 : 1 :: 16 : .225

And consequently $x=4+.225=4.225$, nearly.

Again, if 4.2 and 4.3 be now taken as the two assumed numbers, we shall have

First Sup.		*Second Sup.*
4.2	x	4.3
17.64	x^2	18.49
74.088	x^3	79.507
95.928	Results.	102.297

102.297....4.3..102.297
∴ 95.928....4.2..100
6.369 : .1 :: 2.297 : 036.

And consequently $x=4.3-.036=4.264$, nearly.

Again, let 4.264 and 4.265 be the two assumed numbers. Then

First Sup.		*Second Sup.*
4.264	x	4.265
18.181696	x^2	18.190225
77.526752	x^3	77.581310
99.972448		100.036535

Therefore

100.036535 4.265 100
99.972448 4.264 .99.972448
.064087 : .001 :: .027552 : .0004299, and consequently, $x = 4.264 + .0004299 = 4.2644299$, very nearly.

2. Given $(\frac{1}{5}x^2-15)^2+x\sqrt{x}=90$, to find an approximate value of x.

Here, by a few trials, it will be found that the value of x lies between 10 and 11; which let, therefore, be the two assumed numbers, agreeably to the directions given in the rule.

Then

First Sup.		*Second Sup.*
25	$(\frac{1}{5}x^2-15)$	84.64
31.622	$x\sqrt{x}$	36.482
56.622	Results	121.122

Hence

121.122 .. 11 .. 121.122
56.622 .. 10 .. 90
64.5 : 1 :: 31.122 : .482

And consequently $x=11.-.582=10.518$.

Again, let 10.5 and 10.6 be the two assumed numbers;

Then

First Sup.		*Second Sup.*
49.7025	$(\frac{1}{5}x^2-15)^2$	55.830784
34.0239	$x\sqrt{x}$	34.511099
83.7264	Results	90.341883

Hence

90.341883 .. 10.6 . 90.341883
83.7264 .. 10.5 . 90.
6.615483 .. .1 :: .341883 : .0051679, and consequently $x = 10.6 - .0051679 = 10.5948321$, very nearly.

Of Reciprocal Equations.

104. Although no general method has hitherto been discovered for the resolution of equations higher than those of the fourth power, there are, notwithstanding, some particular equations, of all orders under the 10th, which, on account of the relations subsisting between their coefficients, may be solved by the rules that have been already given for the first four orders.

This is particularly the case with what have been usually called reciprocal equations ; which are such that the coefficients of their terms, taken from the beginning of the equation, are the same as those of the corresponding terms, taken from the end, with the same signs ; or which remain of the same form when the reciprocal of the unknown quantity, or $\frac{1}{x}$, is substituted for x ; except that the terms are then reversed. Thus the equations $x^{2m}+px^{2m-1}+qx^{2m-2}+\ldots\ldots+qx^2+px+1=0$, and $x^{2m+1}+px^{2m}+qx^{2m-1}+\ldots\ldots+qx^2+px+1=0$,may always be transformed into others of a degree which is denoted by half the exponent of the highest power of the unknown quantity, if it be an even number, or by half that exponent when it is diminished by 1, if it be an odd number, the method of resolving them, as far as to the 9th order, inclusively, being as follows. To this we may add, that the nature of these equations consists, as abovementioned, in their not being changed by substituting $\frac{1}{x}$ for x; from which it follows, that if a be any one of the roots, its reciprocal $\frac{1}{a}$ will also be a root ; and as $+1$, or -1, is always a root of the equation, when the number of its dimensions is odd, it may be readily shown from these circumstances, that every equation of the $2m$th or $2m$th$+1$th order, can be reduced to another of the mth order.

CASE I. *When the index of the highest term is an even number.*

RULE I. If the equation be of the fourth power, as $x^4+px^3+qx^2+px+1=0$, find the two values of z in the equation, $z^2+pz+q-2=0$, and let them be denoted by r and r' ; then the roots of the two quadratic equations, $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the four roots of the proposed equation.

2. If it be of the sixth power, as $x^6+px^5+qx^4+rx^3+qx^2+px+1=0$, find the three values of z in the equation, $z^3+pz^2+(q-3)z+r-2p=0$, and let them be denoted by r. r', r'' ; then the roots of the three quadratics $x^2-rx+1=0$, $x^2-r'x+1=0$, and $x^2-r''x+1=0$, will be the six roots of the proposed equation.

3. And if it be of the eighth power, as $x^8+px^7+qx^6+rx^5+sx^4+rx^3+qx^2+px+1=0$, find the four values of z in the equation, $z^4+pz^3+(q-4)z^2+(r-3p)z+s-2(q-1)=0$, and let them be denoted by r, r', r'', r'''.

Then the roots of the four quadratic equations. $x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, and $x^2-r'''x+1=0$, will be the eight roots of the proposed equation.

Case II. ***When the index of the highest term is an odd number.***

Rule I. If the equation be of the third power, as $x^3+px^2+px+1=0$, where one of its roots is evidently -1, find the value of z in the simple equation $z+p-1=0$, and let it be denoted by r; then the roots of the quadratic equation $x^2-rx+1=0$, will be the other two roots of the proposed equation.

2. If it be of the fifth power, as $x^5+px^4+qx^3+qx^2+px+1=0$, where one of its roots is also -1, find the two values of z in the equation $z^2+(p-1)z+q-p-1=0$, and then let them be denoted by r, and r'; then the roots of the two quadratics $x^2-rx+1=0$, and $x^2-r'x+1=0$ will be the other four roots of the proposed equation.

3. If it be of the seventh power, as $x^7+px^6+qx^5+rx^4+rx^3+qx^2+px+1=0$, where one of the roots is -1, as before; find the the three values of z in the equation $z^3+(p-1)z^2+(q-p-2)z+r-p-q+1=0$, and let them be denoted by r, r', r''; then the roots of the 3 quadratics $x^2-rx+1=0$, $x^2-r'x+1=0$, and $x^2-r''x+1=0$, will be the other 6 roots of the proposed equation.

4. And if it be of the 9th power, as

$$x^9+px^8+qx^7+rx^6+sx^5+sx^4+rx^3+qx^2+px+1=0,$$

where one of the roots is -1, as in the former cases, find the four values of z in the biquadratic equation

$z^4+(p-1)z^3+(q-p-3)z^2+(r-q-2p+2)z+s-r-q+p+1=0$, and let them be denoted by r, r', r'', r''';

then the roots of the four quadratics

$x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, and $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

Note. If an equation of this kind be of an odd number of dimensions, or if the middle term of one of an even number of dimensions be wanting, the same rules will hold when the signs of the terms, taken from the beginning, are + and — alternately. Thus $x^3-px^2+px-1=0$, and $x^4-px^3+px-1=0$, are reciprocal equations, like those above given, except that $+1$ is now one of the roots, instead of -1.

1. Given $x^4+4x^3-19x^2+4x+1=0$, to find the four roots of the equation. Here $p=4$ and $q=-19$; hence, by Case 1,

$$z^2+pz=2-q \text{ becomes } z^2+4z=2+19=21,$$

Where the two values of z are $+3$ and -7;

Consequently $x^2-3x=-1$, and $x^2+7x=-1$,

From the first of which equations $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$,

And from the second $x=(-\frac{7}{2}\pm\frac{3}{2}\sqrt{5})=(-\frac{7}{2}\pm\frac{1}{2}\sqrt{45})$;

Therefore the four roots of the proposed equation are

$$\tfrac{3}{2}+\tfrac{1}{2}\sqrt{5},\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{5},\ -\tfrac{7}{2}+\tfrac{3}{2}\sqrt{5},\text{ and }-\tfrac{7}{2}-\tfrac{3}{2}\sqrt{5}.$$

2. Given $x^5-11x^4+17x^3+17x^2-11x+1=0$, to find the five roots of the equation. Here $p=-11$, and $q=17$; hence, by case 2, $z^2+(p-1)z=1+p-q$ becomes $z^2-12x=-27$, where the two values of z are 9 and 3; $x^2-9x=-1$, and $x^2-3x=-1$; from the first of which equations $x=\frac{9}{2}\pm\frac{1}{2}\sqrt{77}$, and from the second $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$. Therefore the five roots of the proposed equation are -1, $\frac{9}{2}+\frac{1}{2}\sqrt{77}$, $\frac{9}{2}-\frac{1}{2}\sqrt{77}$, $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{3}{2}-\frac{1}{2}\sqrt{5}$.

105. If the coefficients of either of the orders of equations, mentioned in the rule, be in part literal, and so constituted as to render the different terms homogeneous, its roots may be determined by means of the simple substitution of a new unknown quantity, as if they were entirely numeral.

Thus, let there be taken, as an equation of this kind, $x^4+4ax^3-19a^2x^2+4a^3x+a^4=0$, where the indices of the given and unknown quantities, when added together, are the same as those of the first and last terms. Then, by putting $x=az$, we shall have $a^4z^4+4a^4z^3-19a^4z^2+4a^4z+a^4=0$, or, dividing by a^4, $z^4+4z^3-19z^2+4z+1=0$; which equation, like the first of those given above, will have for its roots $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, $\frac{3}{2}-\frac{1}{2}\sqrt{5}$, $-\frac{7}{2}+\frac{3}{2}\sqrt{5}$, and $-\frac{7}{2}-\frac{3}{2}\sqrt{5}$; and the four roots of the proposed equation are $a(\frac{3}{2}+\frac{1}{2}\sqrt{5})$, $a(\frac{3}{2}-\frac{1}{2}\sqrt{5})$, $a(-\frac{7}{2}+\frac{3}{2}\sqrt{5})$, and $a(-\frac{7}{2}-\frac{3}{2}\sqrt{5})$.

An homogeneous equation, containing only two letters, a and x, is of the general form

$$x^m+pax^{m-1}+qa^2x^{m-2}+ra^3x^{m-3}+\ldots+ka^{m-1}x+la^m=0,$$

where p, q, r, k, l, are the numeral coefficients.

1. Given $x^4-15x^3+38x^2-15x+1=0$, to find the four roots of the equation. Ans. $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$, and $6\pm\sqrt{35}$.

Here $p=-15$, $q=38$, we have $z^2-15z=-36$, by case 2; $z=\frac{9}{2}\pm\frac{15}{2}=12$ or -3. Hence we have $x^2-12x=-1$, and $x^2+3x=-1$; $\therefore$ in the first we have $x=6\pm\sqrt{35}$, and in the second case we have $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

2. Given $x^4-4ax^3+5a^2x^2-4a^3x+a^4=0$, to find the four roots of the equation. Ans. $a(\frac{3}{2}\pm\frac{1}{2}\sqrt{5})$, and $a(\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$.

Let $p=-4$, $q=5$; then $z^2-4z=2-5=-3$; $\therefore z=3$ or 1, and we have $x^2-3y=-1$, and $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

Again, $x^2-x=-1$; $\therefore\ x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$; consequently we have $a(\frac{3}{2}\pm\frac{1}{2}\sqrt{5})$, and $a(\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$.

3. Given $x^5-21x^4+37x^3+37x^2-21x+1=0$, to find the five roots of the equation. Ans. -1, $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$, and $\frac{19}{2}\pm\frac{1}{2}\sqrt{357}$.

Here $p=-21$, and $q=37$; then by case 2, $z^2-22z=-57$; $\therefore\ z=11\pm8=19$ or 3, and $\therefore\ y^2-19x=-1$; then $x=\frac{19}{2}\pm\frac{1}{2}\sqrt{357}$. Again $x^2-3x=-1$, we find $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

106. Equations of this kind, in which the given and the un-

known quantities can be substituted, alternately, for each other, without producing a new equation, are always capable of being reduced to others of lower dimensions. In order to such a reduction, let the equation, if it be of an even dimension, be first divided by the equal powers of its two quantities in the middle term: then assume a new equation, by putting some quantity (or letter) equal to the sum of the two quotients that arise by dividing those quantities one by the other, alternately; by means of which equation let the said quantities be exterminated; whence a numeral equation will emerge, of half the dimensions with the given literal one.

But, if the equation propounded be of an odd dimension, let it be first divided by the sum of its two quantities, so will it become of an even dimension, and its resolution will therefore depend upon the preceding rule.

Let there be given the equation $x^4-4ax^3+5a^2x^2-4a^3x+a^4=0$. Here, dividing by a^2x^2, we have $\frac{x^2}{a^2}-\frac{4x}{a}+5-\frac{4a}{x}+\frac{a^2}{x^2}=0$, (or $\frac{x^2}{a^2}+\frac{a^2}{x^2}-4\times\frac{x}{a}+\frac{a}{x}+5=0$, by joining the corresponding terms;) and by making $z=\frac{x}{a}+\frac{a}{x}$, and by squaring both sides, we have also $z^2=\frac{x^2}{a^2}+2+\frac{a^2}{x^2}$, or $z^2-2=\frac{x^2}{a^2}+\frac{a^2}{x^2}$. Therefore, by substituting these values, our equation becomes $z^2-2-4z+5$, or $z^2-4z=-3$; whence $z=3$. But $\frac{x}{a}+\frac{a}{x}$ being $=z$, we have $x^2-zax=-a^2$; and consequently $x=\frac{1}{2}za\pm\sqrt{(\frac{1}{4}a^2z^2-a_2)}=\frac{1}{2}a\times\{z\pm\sqrt{(z_2-4)}\}$ $=\frac{1}{2}a\times\{3\pm\sqrt{(5)}\}$, in the present case.

2. Let there be given $x^5+4ax^4-12a^2x^3-12a^3x^2+4a^4x+a^5=0$. In this case, we must first divide by $x+a$, and the quotient will come out $x^4+3ax^3-15a^2x^2+3a^3x+a^4-0$: whence, by proceeding as in the former example, we have $\frac{x^2}{a^2}+\frac{a^3}{x^2}+3\times\frac{x}{a}+\frac{a}{x}-15=0$, or $z^2-2+3z-15=0$, and from thence $z=\frac{\sqrt{(77)}-3}{2}$.

Given $7x^6-26ax^5-26a^5x+7a^6=0$, which, divided by a^3x^3, becomes $7\times(\frac{x^3}{a^3}+\frac{a^3}{x^3})-26\times(\frac{x^2}{a^2}+\frac{a^2}{x^2})=0$. Now making, as before, $z=\frac{x}{a}+\frac{a}{x}$, we have $z^2-2=\frac{x^2}{a^2}+\frac{a^2}{x^2}$; and mult. again by $z=\frac{x}{a}+\frac{a}{x}$, we likewise have have $z^3-2z=\frac{x^3}{a^3}+\frac{a}{x}+\frac{x}{a}+\frac{a^3}{x^3}=\frac{x^3}{a^3}$

$=\frac{x^3}{a^3}+z+\frac{a^3}{x^3}$; and therefore $z^3-3z=\frac{x^3}{a^3}+\frac{a^3}{x^3}$: which values being substituted above, our equation becomes $7\times(z^3-3z)-26\times(z^2-2)=0$, or $7z^3-26z^2-21z+52=0$. Where, trying the divisors of the last term, which are 1, 2, 4, 13, &c. the third is found to answer; z, consequently, being $=4$.

4. Wherein let there be given $2x^7-13a^2x^5-13a^5x^2+2a^7=0$.

Here dividing first by $x+a$, the quotient will be $2x^6-2ax^5-11a^2x^4+11a^3x^3-11a^4x^2-2a^5x+2a^6=0$; which, divided again by a^3x^3, gives $2\times(\frac{x^3}{a^3}+\frac{a^3}{x^3})-2\times(\frac{x^2}{a^2}+\frac{a^2}{x^2})-11\times(\frac{x}{a}+\frac{a}{x})+11=0$, that is, $2\times(z^3-3z)-2\times(z^2-2)-11z+11=0$, or $2z^3-2z^2-17z+15=0$; whence $z=3$.

107. A literal equation may be made to correspond with a numeral one, by substituting an unit in the room of the given quantity (or letter:) and equations that do not seem, at first, to belong to the preceding class, may sometimes be reduced to such by a proper substitution; that is, by putting the quotient of the first term divided by the last, equal to some new unknown quantity (or letter) raised to the power expressing the dimension of the equation. Thus, if the equation given be $2x^4+24x^3-315x^2+216x+162=0$; by putting $\frac{2x^4}{162}=y^4$, we have $x=3y$; whence, after substitution, the given equation becomes $162y^4+648y^3-2835y^2+648y+162=0$; which now answers to the rule, and may be reduced down to $2y^4+8y^3-35y^2+8y+2=0$.

4. Given $7x^6-26x^5-26x+7=0$, to find the roots of the equation. Ans. The only real roots are $2\pm\surd3$.

5. Given $2x^7-13a^2x^5-13a^5x^2+2a^7=0$, to find the roots of the equation. Ans. The only reals roots are
$-a$, $a(\frac{3}{2}\pm\frac{1}{2}\surd5)$, and $-\frac{a}{4}\{(2+\surd14)\pm\surd(2+4\surd14)\}$.

108. *Of Binomial Equations, or such as are of the form* $x^m\pm a^m=0$.

Equations of this kind, which are a peculiar species of reciprocal equations, may be reduced to a more simple form by putting $x=ay$, and then dividing the result by a^m; in which case we shall have $a^my^m\pm a^m=0$, or $y^m\pm1=0$, where the several values of y are the roots of -1, or $+1$; and consequently those of x are the same roots multiplied by a.

Whence, as the first of these forms, $x^m+1=0$, or $x^m=-1$, (using x instead of y,) is a reciprocal equation, wanting all the intermediate terms, its solution may be obtained from the rules before given for this purpose, by making the coefficients, p, q, r, s_1 each equal to 0, and finding the result accordingly.

It therefore only remains to determine the several roots of the equation $x^m-1=0$, or $x^m=1$, which, for the first ten orders, may be done as follows.

Case I. *When the index of the first term is an even number.*

Rule I. If the equation be of the fourth power, as $x^4-1=0$, where two of the roots are, evidently, $+1$ and -1, find the two value of x in the equation $x^2+1=0$, and they will be the other two roots of the proposed equation.

2. If it be of the sixth power, as $x^6-1=0$, where two of the roots are 1 and -1, as before; find the two values of x in each of the equations $x^2+x+1=0$, and $x^2-x+1=0$, and they will be the other four roots of the proposed equation.

3. If it be of the eigth power, as $x^8-1=0$, where two of the roots are also 1 and -1, find the two values of z in the equation $z^2-21=0$, or $z^2=21$, and let them be denoted by r and r'; then the roots of the 3 quadratics $x^2+1=0$, $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the other six roots of the proposed equation.

4. And if it be of the tenth power, as $x^{10}-1=0$, where two of the roots are 1 and -1, as they are in all even powers, find the four roots of the equation $z^4-3z^2+1=0$, and let them be denoted by r, r' r'', r'''; then the roots of the four quadratics $x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, & $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

Case II. *When the index of the first term is an odd number.*

Rule I. If the equation be of the third power, as $x^3-1=0$, where one of the roots is evidently 1; find the two values of x in the equation $x^2+x+1=0$, and they will be the other two roots of the proposed equation.

2. If it be of the fifth power, as $x^5-1=0$, where one of its roots is 1, as before; find the two values of z in the equation $z^2+z-1=0$, and let them be denoted by r and r'; then the roots of the two quadratics $x^2-rx+1=0$, and $x^2-r'x+1=0$, will be the other four roots of the proposed equation.

3. If it be of the seventh power, as $x^7-1=0$, where one of its roots is also 1; find the three values of z in the equation $z^3+z^2-2z-1=0$, and let them be denoted by r, r', r''; then the roots of the 3 quadratics, $x^2-rx+1=0$, $x^2-r'x+1=0$, & $x^2-r''x+1=0$, will be the other six roots of the proposed equation.

4. And if it be of the ninth power, as $x^9-1=0$, where one of its roots is likewe 1, as in all the odd powers, find the four values of z in the equation $z^4+z^3-3z^2-2z+1=0$, and let them be denoted by r, r' r'' r'''; then the roots of the four quadratics

$x^2-rx+1=0$, $x^2-r'x+1=0$, $x^2-r''x+1=0$, & $x^2-r'''x+1=0$, will be the other eight roots of the proposed equation.

109 Since two of the values of x, in every case of even powers, are $+1$ and -1, if the first member of an equation of the form $x^{2m}-1=0$, be divided by x^2-1, the quotient will be

$$x^{2m-2}+x^{2m-4}+x^{2m-6}+\ldots+x^4+x^2+1=0,$$

And because one of the values of x, in every case of the odd powers, is 1, if the first member of an equation of the form $x^{2m+1}-1=0$,be divided by $x-1$, the quotient will be

$$x^{2m}+x^{2m-1}+x^{2m-2}+x^{2m-3}+\ldots+x^2+x+1=0,$$

both of which are reciprocal equations, having all the coefficients of their several terms $=1$; and, consequently, each of them may be resolved by the method employed in the last article, without recurring to any separate considerations.

1. Required to find the four roots of the equations $x^4-1=0$.

Here, the index of x being an even number, two of the roots are $+1$ and -1. by *Case* I, we have $x^2+1=0$, or $x=\pm\sqrt{-1}$; whence the four roots are 1, -1, $+\sqrt{-1}$, and $-\sqrt{-1}$.

2. It is required to find the three roots of the equation $x^3-1=0$.

Here, the index of x being an odd number, one of the root is 1.

And, by Case II, we shall have $x^2+x+1=0$, or $x^2+x=-1$. Whence, by quadratics, $x=-\frac{1}{2}\pm\sqrt{(\frac{1}{4}-1)}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

$\therefore$ 1, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, are the roots sought.

And if it be required to find the three roots of the equation $x^3+1=0$, we shall have, in this case, one of the roots $=-1$, And, by dividing x^3+1 by $x+1$, there will arise $x^2-x+1=0$,or $x^2-x=-1$ Consequently,$x=\frac{1}{2}\pm\sqrt{(\frac{1}{4}-1)}=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$. that is -1, $\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, are the roots sought.

3. It is required to find the five roots of the equation $x^5-1=0$.

Here 5 being an odd power, 1 root of the given equation is $=1$

Therefore, by the rule, Case II, $z^2+z-1=0$, or $z^2+z=1$;

$z=-\frac{1}{2}+\sqrt{(\frac{1}{4}+1)}=-\frac{1}{2}+\frac{1}{2}\sqrt{5}=r$, and $z=-\frac{1}{2}-\frac{1}{2}\sqrt{5}=r'$.

Whence, also, $x^2-rx+1=0$, or $x^2-(-\frac{1}{2}+\frac{1}{2}\sqrt{5})x=-1$. And,

$x=\frac{1}{4}(-1+\sqrt{5})\pm\sqrt{\{\frac{1}{16}(-1+\sqrt{5})^2-1\}}=\frac{1}{4}(-1+\sqrt{5})$
$\pm\frac{1}{4}\sqrt{(-10-2\sqrt{5}}=\frac{1}{4}(-1+\sqrt{5})\pm\frac{1}{4}(\sqrt{10+2\sqrt{5}})\sqrt{-1}$.

In like manner, $x^2-r'x+1=0$, or $x^2+(\frac{1}{2}+\frac{1}{2}\sqrt{5})x=-1$, where $x=-\frac{1}{4}(1+\sqrt{5})\pm\sqrt{\{\frac{1}{16}(1+\sqrt{5})^2-1\}}=-\frac{1}{4}(1+\sqrt{5})\pm\frac{1}{4}(\sqrt{10-2\sqrt{5}})\sqrt{-1}$. Therefore the five roots are

$$1,\ \tfrac{1}{4}(-1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{10+2\sqrt{5}})\sqrt{-1},\ \text{and}$$
$$-\tfrac{1}{4}(1+\sqrt{5})\pm\tfrac{1}{4}(\sqrt{10-2\sqrt{5}})\sqrt{-1}.$$

1. It is required to find the four roots of the equaton $x^4+1=0$.

See Index. Ans. $x=\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2}$, and $-\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2}$.

2. It is required to find the five roots of the equation $x^5+1=0$.

Ans. -1, $\frac{1}{4}(1+\sqrt{5})\pm\frac{1}{4}(\sqrt{10-2\sqrt{5}})\sqrt{-1}$, and $\frac{1}{4}(1-\sqrt{5})\pm\frac{1}{4}(\sqrt{10+2\sqrt{5}})\sqrt{-1}$.

3. It is required to find the six roots of the equation $x^6-a^6=0$.

Ans. a, $-a$, $a(-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$, and $a(\frac{1}{2}\pm\frac{1}{2}\sqrt{-3})$.

4. It is required to find the eight roots of the equation $x^8-1=0$.

Ans. 1, -1, $\sqrt{-1}$, -1, $(\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2})$, $(-\frac{1}{2}\sqrt{2}\pm\frac{1}{2}\sqrt{-2})$.

5. Find the roots of the equations $x^{10}-a^{10}=0$, and $x^{15}-1=0$.

Of Equations that have equal Roots.

110. Besides the classes of equations before treated of, there are others of a different kind, that are equally susceptible of being reduced to those of lower dimensions; the most useful of which are such as have two or mose equal roots, with the same or contrary signs; in which case the method of resolving them, as far as the 4th order, inclusively, are as follows:*

RULE I. If a quadratic equation of the general form $x^2\pm ax+b=0$, has two equal roots, with the same sign, they will be each $=-\frac{1}{2}a$, or $+\frac{1}{2}a$, according as the coefficient of the second term is positive or negative. And if the equal roots have contrary signs, the equation must be of the form $x^2-b^2=0$, in which case $x=\pm\sqrt{b^2}$, or $=+b$, and $-b$.

2. If a cubic equation of the general form, $x^3+ax^2+bx+c=0$, has two equal roots, with the same sign, each of them will be a root of the equation $3x^2+2ax+b=0$, and the remaining root will be $=-a-$ twice one of the equal roots. And if the equal roots have contrary signs, we shall have in that case, $ab=c$; and the roots sought will be $\sqrt{-\frac{c}{a}}$, and $-\sqrt{-\frac{c}{a}}$, and the remaining root will be $=a$.

3. If a biquadratic equation of the form $\left.\begin{array}{l} x^4+ax^3+bx^2+cx+d=0, \\ \;\; -38 \;\; 210 \;\;\; 538 \;\; 289, \end{array}\right\}$ has two equal roots, with the same sign, each of them will be a root of the equation

$(3a^2-8b)x^2+2(ab-6c)x+ac-16d=0$; $x^2-\frac{22416}{2652}x=\frac{25068}{2652}$

$x^2-\frac{22416}{2652}x=\frac{25058}{2652}$, or $x^2-\frac{1868}{221}x=\frac{2089}{221}$. By Case II,

$x^2-\frac{1868}{221}x+(\frac{934}{221})^2=(\frac{2089}{221}+\frac{872356}{48841})=\frac{1334025}{48841}$; $x-\frac{934}{221}=$

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* Other methods of resolving equations that have equal roots, have been given by *Maclaurin*, and several of the best foreign writers on Algebra, which, though not so simple as those here laid down, are more commodious for equations of the higher orders, which, if treated in the manner employed in the text, would become very tedious.

$\sqrt{\frac{1334025}{48841}}=\frac{1155}{221}$, or $x=\pm\frac{1155}{221}+\frac{934}{221}=-\frac{221}{221}$, or -1, and the other two roots will be found from the equation

$$x^2+(a+2r)x+b+2ar+3r^2=9, \text{ or } x^2-40x=-289,$$

r being one of the equal roots before obtained.

And if the equal roots have contrary signs, we shall have $a(bc-ad)=c^2$, and $x=\pm\sqrt{-\frac{c}{a}}$ as before; the other two roots being found from the equation $x^2+ax+b+r^2=0$.

4. Also, if the general biquadratie equation $x^4+ax^3+bx^2+cx+d=0$, has three equal roots, with the same sign, each of them will be a root of the equation $6x^2+3ax=-b$. And if two of the three equal roots have contrary signs, we shall have

$$a(bc-ad)x=c^2, \text{ and } x=\pm\sqrt{-\frac{c}{a}}, \text{ as in the last case.*}$$

1. Supposing that two of. the roots of the equation $x^3+x^2+33+63=0$, are equal to each other, and have the same sign, it is required to determine them.

Here a being $=1$, $b=-33$, and $c=63$, we shall have, by the rule above laid down, $3x^2+2ax+b=3x^2+2x-33=0$; or $x^2-\frac{2}{3}x=11$; whence $x=-\frac{1}{3}\pm\sqrt{(\frac{1}{9}+11)}=-\frac{1}{3}\pm\sqrt{\frac{100}{9}}=-\frac{1}{3}\pm\frac{10}{3}=3$, or $=-\frac{1}{3}-\frac{10}{3}=-\frac{11}{3}$, the former of which values (3) being substituted for x, in the original equation, is found to succeed.

Whence two of the three roots are each $=3$; and the remaining root $=-7$.

2. Supposing that two of the roots of the equation $x^4+\frac{1}{2}x^3-\frac{3}{2}x^2+\frac{4}{27}x+\frac{2}{9}=0$, are equal to each other, and have the same sign; it is required to determine them.

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* The mode of solution here given might have been easily extended to equations of the 5th, 6th, &c. power; but in those above the 4th degree, the expressions for the equal roots, when they have the same sign, are too complicated to be of any practical use.

Rules might have been also introduced for determining, *a priori*, by meams of the relations of the coefficients alone, whether or not any given equation has equal roots; but, except in the case where they have contrary signs, this may in general be more readily done, by finding the root of the quadratic equation to which it is reducible, and then seeing whether the root, thus obtained, be a root of the original equation or not.

In all cases of this kind, however, it may be of use to know, that an equation cannot have equal roots when the last term, and the coefficient of the last but one, are prime to each other; but the reverse of this is not always true.

Here a being $=\frac{1}{2}$, $b=-\frac{3}{2}$, $c=\frac{4}{27}$, and $d=\frac{2}{9}$, we shall have, by the rule, $(3a^2-8b)x^2+2(ab-6c)x+ac-16d=\frac{51}{4}x^2-\frac{59}{18}x-\frac{94}{27}=0$. Or, by transposition and simplifying the terms, $x^2-\frac{118}{459}x=\frac{376}{1377}$; the two roots of which equation are $\frac{2}{3}$ and $=\frac{188}{459}$.

And by trial, the first of these, $\frac{2}{3}$, will be found to be a root of the original equation. Whence thetwo roots required are each $=\frac{2}{3}$.

And, since $r=\frac{2}{3}$, we shall have, by the rule,

$x^2+(a+2r)x+b+2ar+3r^2=x^2+(\frac{1}{2}+\frac{4}{3})x-\frac{3}{2}+\frac{2}{3}+\frac{4}{3}=0$, or $x^2+\frac{11}{6}x=-\frac{1}{2}$; the roots of which latter equation are $-\frac{1}{3}$, $-\frac{3}{2}$. Hence the four roots of the given equation are $\frac{2}{3}$, $\frac{2}{3}$, $-\frac{1}{3}$, and $-\frac{3}{2}$.

3. It is required to determine whether the equation $x^4+11x^3+19x^2-99x-252=0$, has equal roots, and if so, what they are.

Here, since $a=11$, $b=19$, $c=-99$, and $d=-259$, we shall have, by the rule $a(bc-ad)=11(-19\times99+11\times252)=11(-1881+2772)=9801=99^2=c^2$; whence the equation has two equal roots, with contrary signs; and, consequently,

$$x=\pm\sqrt{-\frac{c}{a}}=\pm\sqrt{\frac{99}{11}}=\pm\sqrt{9}=3, \text{ and } -3,$$

which two roots will be found to answer the conditions of the question. Also, since $r=\pm3$, we shall have, by the rule, $x^3+ax+b+r^2+=x^2+11x+28=0$, or $x^2+11x=-28$; the roots of equations are -4 and -7 W ;hence the four roots of the given equation are 3, -3, -4, and -7.

1. Supposing that two roots of the equation $x^3-48x-128=0$, are equal, and have the same signs; it is required to determine them. Ans. -4, -4, and 8.

11 . A cubic equation of the general form $x^3+px^2+qx+r=0$, has two equal roots with the same signs. Let its 3 roots be denoted by a, b, c; then we shall have the two following equations; $\{a^3+pa^2+qa+r=0\}-\{b^3+pb^2+qb+r=0\}=a^3-b^3+p(a^2-b^2)+q(a-b)=0$; and when divided by $a-b$, will give $a^2+ab+b^2+p(a+b)+q=0$. If we now take $b=0$, the reduced equationin this case will be $3a^2+2pa+q=0$, or $=3x^2+2px+q=0$, where one of the values of x, taken twice with the same sign, will give the two equal roots of the proposed equation.

2. Given the equation $x^3+5\frac{1}{2}x^2-\frac{1}{16}x-\frac{11}{32}=0$, to find whether it has equal roots, and if so, what they are. Ans. $\frac{1}{4}$, $-\frac{1}{4}$, & $-5\frac{1}{2}$.

3. Supposing the equation $x^4+3a^3-14x^2-12x+40=0$, to have three equal roots, two of which have the same sign; it is required to determine them. Ans. 2, 2, -2, and -5.

4. Supposing the equation $x^4-\frac{51}{400}x^2-\frac{1}{800}x+\frac{3}{800}=0$, to have two equal roots, with thesame sign; it is required to determine all its four roots. Ans. $-\frac{1}{4}$, $-\frac{1}{4}$, $\frac{1}{5}$, and $\frac{3}{10}$.

Of Exponential Equations.

111 An exponential quantity is that which is raised to some unknown power, or which has a variable quantity for its index, as a^x, $a^{\frac{1}{x}}$, x^x, or $a^{\frac{1}{x}}$, &c. See Logarithms.

And an exponential equation is that which is formed between any expression of this kind and some other quantity whose value is known; as $a^x=b$, $x^x=a$, &c.; where it is to be observed, that the first of these equations $a^x=b$, when converted into logarithms, is the same as x log. a=log. b; or $x=\frac{\log. b}{\log. a}$; and the second $x^x=a$, when so converted, is the same as x log. x=log. a; in the last of which cases the approximate value of x may be determined as follows:

Rule. Find, by trial, two numbers nearly equal to the value of x, and substitute them in the given equation, x log. x = log. a, instead of the unknown quantity, noting the results obtained from each, as in the rule of Double Position, before laid down, in art. 31.

Then, by means of a certain number of successive operations, performed in the same manner as is there described, the value of x may be found to any degree of accuracy required.

1. Given $x^x=100$, to find an approximate value of x.

Here, by the above formula, we have x log. x = log. 100 = 2,

And since x is easily found, by a few trials, to be nearly in the middle between 3 and 4, but rather nearer the latter than the former, let 3.5 and 3.6 be taken for the two assumed numbers. Then log. 3.5=.5440680; which, being multiplied by 3.5, gives 1.904238 = first result; And log. 3.6 = .5563025; which being multiplied by 3.6, gives 2.002689 for the second result. Whence 2.002689....3.6....2.002689

1.904238....3.5....2.

.098451 : .1 :: .002689 : .00273

for the first correction; which, taken from 3.6, leaves x = 3.59727, nearly. And as this value is found, on trial, to be rather too small, let 3.59727 and 3.59728 be taken as the two assumed numbers. Then log. 3.59727 = .5559731; which, being multiplied by 3.59727, gives 1.9999854 = first result.

And log. 3.59728 = .5559743; which, being multiplied by 3.59728, gives 1.9999953 = second result; whence

1.9999953....3.59728....2

1.9999854....3.59727....1.9999053

.0000099 : .00001 :: .0000047 : .00000474747

for the second correction; which, added to 3.59728, gives x = 3.59728474747, extremely near the truth.

And in the same way may the value of the unknown quantity be determined, in any other equation of this kind.

1. Given $x^x=2000$, to find an approximate value of x.

Here x log. $x=$ log. $2000 = 3.301030$, and x is found, by a few trials, to be rather less than 5; let therefore 4.8 and 4.9 be taken for the assumed numbers; then log. $4.8=0.681241$, which, being multiplied by 4.8, gives $3.269956 =$ the first result; and log. $4.9 = 0.690196$, which being multiplied by 4.9, gives $3.381960 =$ the second result. Whence

$$\begin{array}{lllll} 3.381960 & \ldots & 4.9 & \ldots & 3.381960 \\ 3.269956 & \ldots & 4.8 & \ldots & 3.301030 \\ \hline .112004 & : & .1 & :: & .080930 \end{array}$$

$: 00722$ for the first correction, which, being taken from 4.9, leaves 4.8278, the answer nearly.

2. Given $(6x)^x=96$, to find the approximate value of x.

Here x log. $6x =$ log. 96, and x is easily found, by a few trials, to be rather less than 2; let therefore 1.8 and 1.9 be taken for the two assumed numbers; then log. $6x(=10.8) = 1.033424$, which, multiplied by 1.8, gives $1.860363 =$ the first result; and log. $6x(= 11.4) = 1.056905$, which, multiplied by 1.9, gives $2.008119 =$ the second result. Whence

$$\begin{array}{lllll} 2.008119 & \ldots & 1.9 & \ldots & 2.008119 \\ 1.860363 & \ldots & 1.8 & \ldots & 1.982271 \\ \hline .147756 & : & .1 & : & .025848 \end{array}$$

$: .0174$ for the first correction, which, taken from 1.9, leaves 1.8826 for the answer,

3. Given $x^x=123456789$, to find the approximate value of x.

Here, after a few trials, or from inspection in a table of powers we find x is between 8 and 9, but nearer the latter than the former. Assume therefore $x=8.6$ and $x=8.7$.

Then log · $123456789 = 8.0915149$

$$\begin{array}{llllll} \text{Log.}8.6=0.9344935 & \text{L.}8.7=0.9395193 & 8.17381 & & 8.7 & 8.09151 \\ \text{Mult. by} \quad 8{\cdot}6 & 8{\cdot}7 & 8{\cdot}03664 & & 8{\cdot}6 & 8{\cdot}03664 \\ \hline 56069610 & 65766351 & {\cdot}13717 & : & 0.1 & :: {\cdot}05487 \\ 7{\cdot}4759480 & 7{\cdot}5161544 & & & & : {\cdot}04 \\ \hline 8{\cdot}03664410 \quad \text{Res} & 8{\cdot}17381791 & & & & \text{Whence} \end{array}$$

$x = 9{\cdot}6 + {\cdot}04 = 8{\cdot}640$ nearly; And repeating the operation by assuming x equal to $8{\cdot}64$ and $8{\cdot}641$; x is found $= 8{\cdot}6400268$

4. Given $x^x-x=(2x-x^x)^{\frac{1}{x}}$, to find the approximate value of x. Ans. $x=1.47933$.

5. Given $a^{mx+n}=b^{nx+m}$, to find x. Ans. $x=\dfrac{\text{log. } b^m-\text{log. } a^n}{\text{log. } a^m-\text{log. } b^n}$.

6. Given $\left.\begin{array}{r} a^{xy}=m \\ b^{x+y}=n \end{array}\right\}$ to find the values of x and y. See Index.

7. Given $x^x=48$, $(\frac{5}{6})^x=\frac{1}{2}$, and $x^x=100$, to find x.

Ans. $x = 3.26$, $x = 3.8$ and $x = 3.5972$.

Here x is found, by a few trials, to be nearly equal to 1.7; let therefore 1.7 and 1.8 be taken for the assumed numbers:

1st Supposition.	*2d Supposition.*
By logs. $x^x = 2.46471$	$x^x = 2.88072$
$x^x - x = 0.76471$	$x^x - x = 1.08072$
$2x - x^x = .93529$	$2x - x^x = .71929$
$\log.(x^x - x) = -1.883496$	$\log.(x^x - x) = 0.033713$
$\frac{1}{x}\log.(2x - x^x) = -1.982909$	$\frac{1}{x}\log.(2x - x^x) = -1.864943$

$$\begin{array}{rcr} 0.033713 & & -1.982909 \\ -1.864943 & \text{results} & -1.883496 \\ \hline .168770 & & .098413 \end{array}$$

Therefore .168770 : .1 :: .098413 : .058. Whence 1.8 — .058 =1.742, the value fo x, nearly; and, assuming 1.74 and 1.75 for the numbers, and repeating the operation, we get $x=1.74793$.

5. Given $a^x + \frac{1}{a^x} = b$, to find the value of x.

$$a^{2x} - ba^x = -1\,;\ a^{2x} - ba^x + \frac{b^2}{4} = \frac{b^2-4}{4}\,;\ a^x = \frac{b \pm \sqrt{(b^2-4)}}{2}\,;$$

$$\therefore x \cdot \log.\ a = \log.\ \frac{b \pm \sqrt{(b^2-4)}}{2} = \log.\ \{b \pm \sqrt{(b^2-4)} - \log.\ 2\,;$$

$$\therefore x = \frac{\log.\ \{b \pm \sqrt{(b^2-4)} - \log.\ 2}{\log.\ a}$$

6. Given $a^{mx} + a^{mx-1} = b$, to find the value of x.

$$a^{mx} + a^{mx-1} = b\,;\ \text{or } a^{mx} + \frac{a^{mx}}{b} = b\,;\ \therefore a^{mx} = \frac{ab}{a+1}.$$

Take the log. and $mx.\ \log.\ a = \log.\ (ab) - \log.\ (a+1)$

$$\therefore x = \frac{\log.(ab) - \log.(1+1)}{m \cdot \log.\ a}.$$

112. In considering the nature of an exponential of the form a^{b^x}, it must be observed that it means a to the power of b^x, and not a^b to the power of x, which would be a^{bx}.

7. Given $x^x = a$, to find the value of x.

Find by trial a number, n, as near to the true value of x as can be done. Let $n+y=x$, then by taking the log. of the given equation,

$$(n+y) \cdot \log.(n+y) = \log.\ a\,;\ \text{i. e. } (n+y) \cdot \log.\ n \cdot (1 + \frac{y}{n}) = \log.\ a,$$

$$\text{Or } (n+y) \cdot \log.\ n + (n+y)\mathrm{M} \left\{ \frac{y}{n} - \frac{y^2}{2n^2} + \&c. \right\} = \log.\ a\,;$$

Then by multiplying $(n+y)$ into the series, and rejecting all the terms into which y^2, y^3, &c. enter, we obtain $(n+y) \cdot \log.\ n +$

$My = \log. a$. Whence $y = \dfrac{\log. a - n \log. n}{\log. n + M}$; and $\therefore x = n + y = n + \dfrac{\log. a - n \cdot \log. n}{\log. n + M} = \dfrac{\log. a + Mn}{\log. n + M}$, which affords a near approximate value of x. If extreme accuracy be required, we must use this value of x in the place of n, and repeat the operation.

Of the method of Indeterminate Coefficients.

113. This is a species of investigation which is frequently used for obtaining the development of certain fractional and other expressions, without having recourse to the operations of division, or the extraction of roots; the method of performing which is as follows:*

Rule. Assume a series, or other expression, with unknown coefficients, for that which is required to be found; then, having multiplied it by the denominator of the given fraction, or raised it to its proper power, find the value of each of these coefficients, by equating the homologous terms of the two expressions, or putting such of them as have no corresponding terms, equal to 0, as the case may require. Find the development of $\dfrac{a}{b + cx}$, according to the above method.

Assume $\dfrac{a}{b + cx} = A + Bx + Cx^2 + Dx^3 + Ex^3 + \&c.$

Then, multiplying the right hand side of the equation by $b + cx$, and transposing a, we shall have

$$0 = \begin{array}{l|l|l|l|l|l} Ab & +Bb & x + Cb & x^2 + Db & x^3 + Eb & x^4 + Fb \\ -a & +Ac & \;\;+Bc & \;\;+Cc & \;\;+Dc & \;\;+Ec \end{array} \, x^5 + \&c.$$

And by putting the first term, and the coefficients of the several powers of $x = 0$, there will arise the following equations:

$$\left.\begin{array}{l} Ab - a = 0 \\ Bb + Ac = 0 \\ Cb + Bc = 0 \\ Db + Cc = 0, \ \&c. \\ \left.\begin{array}{l} Eb + Dc = 0, \\ Fb + Ec = 0, \end{array}\right\} \end{array}\right| \text{ or } \left|\begin{array}{l} A = \frac{a}{b};\ B = -\frac{c}{b}A \\ C = -\frac{c}{b}B, \ \&c. \ D = -\frac{c}{b}C; \\ E = \frac{a}{b}D; \quad F = \frac{a}{b}E \end{array}\right.$$

Hence $\dfrac{a}{b + cx} = \dfrac{a}{b} - \dfrac{c}{b}Ax - \dfrac{c}{b}Bx^2 - \dfrac{c}{b}Cx^3 - \dfrac{c}{b}Dx^4 - \&c.$

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* The method here described of assuming certain expressions, with indeterminate coefficients, in the investigation of analytical problems, and then determining their values by means of their relations to other quantities that are known, appears to have been first employed by *Descartes*, who obtained by this means the rule for revolsing biquadratic equations, which still goes by his name.

where it is obvious, that each coefficient, after the second is equal to the one that immediately precedes it, multiplied by $-\frac{c}{b}$ which law renders it unnecessary to take a greater number of equations, or to push the calculation farther.

2. Required the development of $\frac{d+ex}{a'+b'+c'x^2}$, according to the [above method.

Assume $\frac{d+ex}{a'+b'x+c'x^2}=A+Bx+Cx^2+Dx^3+Ex^4+$ &c.

Then multiplying the right hand side of the equation by $a+b'x+c'x^2$, and transposing $a+bx$, we shall have

$$\begin{array}{r|r|r|r|l} 0=Aa'+Ba' & x+Ca & x^2+Da' & x^3+Ea & x^4+\text{ \&c.} \\ -\ d+Ab' & +Bb' & +Cb' & +Db & \\ -e & +Ac' & +Bc' & +Cc & \end{array}$$

And, by putting the first term, and the coefficients of the several powers $=0$, there will arise

$$\left.\begin{array}{r} Aa'-d=0 \\ Ba'+Ab'-e=0 \\ Ca'+Bb'+Ac'=0 \\ Da'+Cb'+Bc'=0, \end{array}\right| \begin{array}{c} \\ \text{or} \\ \\ \text{\&c.} \end{array} \left|\begin{array}{l} A=\frac{d}{a'};\ B==\frac{b'}{a'}A+\frac{e}{a'} \\ C=-\frac{b'}{a'}B-\frac{b'}{a'}A \\ \\ D=-\frac{b'}{a'}C-\frac{c'}{a'}B\ \text{\&} \end{array}\right.$$

Lastly $\quad Ea+Db+Cc=0;\ \therefore\ E=-\frac{b}{a}D-\frac{c}{a}C.$

Whence $\frac{d+ex}{a'+b'x+c'x^2}=\frac{a}{a'}-(\frac{'d}{'a}A-\frac{e}{a'})x,-(\frac{b'}{a'}B-\frac{c'}{a'}A)x^2-(\frac{b'}{a'}C+\frac{c'}{a'})x^3$—&c., where each coefficient, beginning with the third, may be readily deduced from the two that precede it.

So that if if P, Q, R, be any three consecutive coefficients, we shall have $Ra'+Qb'+Pc'=0$; or $R=-\frac{b'}{a'}Q-\frac{c'}{a'}P$.

3. Given $(x^2+p)^2-(qx+r)^2=x^4+ax^2+bx+c$, to find the indefinite coefficients p, q, and r.

Here, by squaring the terms on the left hand side of the equation, and collecting those that are alike, we have

$$x^4+(2p-q^2)x^2-2qrx+p^2-r^2=x^4+ax^2+bx+c.$$

And consequently, by equating the homologous terms,

$$\left.\begin{array}{r} 2p-q^2=a \\ -2qr=b \\ p^2-r^2=c \end{array}\right| \text{ or } \left|\begin{array}{l} 2p-a=q^2 \\ -b=2qr \\ p^2-c=r^2 \end{array}\right.$$

where it is plain, that the product of the first and third of these is equal to $\frac{1}{4}$ of the square of the second; or, $2p^3-ap^2-2cp+ac=\frac{1}{4}b^2$.

Whence the value of p may be found by a cubic equation, and then q and r from the former equations.

We shall further observe, that every algebraic function, containing one unknown quantity, or that quantity and its powers, may be assumed of the following form: $A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+$ &c., of which the terms involve only the whole positive powers of the unknown quantity x, and certain coefficients A_0, A_1, A_2, &c. which are wholly independent of the value of that quantity.

114. This may be exemplified by considering the expansions of such functions as usually occur; where the particular series that is to be adopted, in each case, may be determined, by putting $x=0$ and observing the nature of the result.

1. Thus, if the function to be developed is to become = to some particular quantity for $x=0$, the first term of the series must be taken singly, or without an x; and if it is to become equal to zero, for $x=0$, the first term must contain x.

2. Also, if any term be either omitted or introduced into the series that does not belong to it, the comparisons of the homologous terms will always show the impropriety of it, by their leading to absurd equations. Thus taking, for instance, as a partial example of the first of these cases $\frac{1}{1+x}=A_0+A_1x+A_2x^2+A_3x^3+$&c., is plain, that when $x=0$, we shall have $\frac{1}{1}=A_0$, or $A_0=1$; which is therefore the first term of the series.

3. In like manner, if we take the expansion of a fraction of the following kind, $\frac{x}{1+x}=A_1x+A_2x^2+A_3x^3+A_4x^4+$&c. it is equally evident that the first term of the series must contain x, for $x=0$; for, if it were A_0, $\frac{0}{1}$ would be equal to some determinate quantity, which is impossible.

4. Also, if there be taken, as an example of the last of the above cases, $\frac{1}{1-x}=A_0+A_1x \cdots + A_3x^3+A_4x^4+$ &c., where the term A_2x^2 of the series is omitted, it will be found, by multiplying the right hand member of the identity by $1-x$, and equating the homologous coefficients, that $A_0=1$, and $A_1=1$, as also $A_1=0$; which is absurd.

5. Again, let there be taken, as an additional example, the well known formula $\frac{x^m-v^m}{x-v}=x^{m-1}+A_2x^{m-2}+A_3x^{m-3}+\dots+v^{m-1}$; where, supposing m to be a whole positive number, it is evident that the series is rightly assumed; because, for $x=0$, the development is reduced to v^{m-1}, and for $v=0$ it is reduced to x^{m-1}, as

it ought.* Hence, multiplying the right hand member by $x - v$ we shall have

$$x^m - v^m = \begin{array}{l|l|l|l} x^m + A_2 & x^{m-1} + A_3 & x^{m-2} + A_4 & x^{m-3} \ldots + v^{m-1} x \\ \quad\;\; - v & \qquad\;\; - A_2 v & \qquad\;\; - A_3 v & \qquad \ldots - v^m \end{array}$$

6. And, consequently, by putting the coefficients of the several terms of the product, except the first and last, equal to 0, there will arise the following relations :

$$\begin{array}{l|c|l} A_2 - v = 0 & & A_2 = v \\ A_3 - A_2 v = 0 & \text{or} & A_3 = v^2 \\ A_4 - A_3 v = 0 & & A_4 = v^3 \\ A_5 - A_4 v = 0, \text{ \&c.} & & A_5 = v^4, \text{ \&c.} \end{array}$$

From which it is evident, that the proposed formula, when expanded, will become

$$\frac{x^m - v^m}{x - v} = x^{m-1} + v x^{m-2} + v^2 x^{m-3} + v^3 x^{m-4} + \ldots\ldots\ldots + v^{m-1}$$

115. In like manner, supposing, when m is a whole number, the following polynomial, or series,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots\ldots\ldots + Tx + V,$$

to be exactly divisible by $x - a$, we shall have, by assuming a certain series, $x^{m-1} + A'x^{m-2} + B'x^{m-3} +$ &c. for the quotient:

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + V =$$
$$(x - a)(x^{m-1} + A'x^{m-2} + B'x^{m-3} + \ldots \quad + S'x + T');$$

where A, B, C, &c. are supposed to be known coefficients, and A′, B′, C′, &c. other coefficients that are to determined.

Then, by actually performing the multiplication indicated by the second member of the identity, there will arise,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots\ldots Tx + V$$

$$= \left|\begin{array}{l|l|l|l|l|l} x^m + A' & x^{m-1} + B' & x^{m-2} + C' & x^{m-3} + S' & x^2 + T' & x - aT' \\ \qquad - a & \qquad - aA' & \qquad - aB' & \qquad - aR' & \quad + aS' & \end{array}\right.$$

In which case, as the corresponding powers of x in both series has a term, that for the development of the quotient appears to have been rightly assumed. Wherefore, bringing all the terms to the left hand side of the equation, we shall have

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* In the case where the index, in the above expression, is a fractional number, which requires a different investigation, the same formula may be exhibited under the form

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n} - 1} \left| \frac{1 + \left(\frac{x}{v}\right) + \left(\frac{x}{v}\right)^2 + \left(\frac{x}{v}\right)^3 + \ldots\ldots + \left(\frac{x}{v}\right)^{m-1}}{+1 \left(\frac{x}{a}\right)^{\frac{m}{n}} + \left(\frac{x}{v}\right)^{\frac{2m}{n}} + \left(\frac{x}{v}\right)^{\frac{3m}{n}} + \ldots + \left(\frac{x}{v}\right)^{\frac{(m-1)m}{n}}} \right.$$

being, in this state, a similar theorem to that given by *Landen*, in his *Residual Analysis*, from which he attempted to derive the Doctrine of Fluxions, now the method of the Calculis.

$$\begin{array}{r|r|r|r|r|l} A & x^{m-1}+B & x^{m-2}+C & x^{m-3}+\ldots+R & x^2+T & x+V=0 \\ -A_1 & -B_1 & -C' & -R' & -T' & -aT \\ +a & +aA' & +aB' & +aT' & +aS' & \end{array}$$

And, consequently, by equating the coefficients of the several powers of x with 0, there will arise the following relations;

$$\left|\begin{array}{l} A-A_1+a=0 \\ B-B_1+aA'=0 \\ C-C'+aB'=0 \\ D-D'+aC'=0,\ \&c. \end{array}\right| \text{ or } \left|\begin{array}{l} A'=a+A \\ B'=a^2+aA+B \\ C'=a^3+a^2A+aB+C \\ D'=a^4+a^3A+a^2B+aC+D,\ \&c. \end{array}\right.$$

where the law by which they may be continued being sufficiently evident, we shall have, for the quotient,

$$\begin{array}{lr|lr|lr|ll} Q=x^{m-1} & +a & x^{m-2} & +a^2 & x^{m-3} & +a^3 & x^{m-4}+\ldots & +a^{m-1} \\ & +A & & +aA & & +a^2A & & +a^{m-2}A \\ & & & +B & & +aB & & +a^{m-3}B \\ & & & & & +C & & +a^{m-4}C\ \&c. \end{array}$$

And as the process here followed is the same in effect as that which would have taken place from a direct comparison of the corresponding terms of the two members of the identity, it follows, that, when two polynomials are identical, the coefficients of the respective powers of the unknown quantity, in each of them, will be equal.

So that, from these instances, and others of a like kind, that might be given, the form of the series for any expanded algebraic function, and the values of the assumed coefficients may always be determined.

4. Find the expansion of the expression $\frac{a}{b-cx}$. Assume $\frac{a}{c-cx}=A+Bx+Cx^2+Dx^3+Ex^4+\&c.$ By freeing this equation of fractions

$$a=\begin{array}{r|r|r|r|l} Ab+Bb & x+Cb & x^2+Db & x^3+Eb & x^4+\&c. \\ -Ac & -Bc & -Cc & -Dc & +\&c. \end{array}$$

By transposition,

$$\left.\begin{array}{r|r|r|r|l} Ab+Bb & x+Cb & x^2+Db & x^3+Eb & x^4+\&c. \\ -a-Ac & -Bc & -Cc & -Dc & -\&c. \end{array}\right\}=0;$$

$$\text{Therefore } Ab-a=0;\ \therefore Ab=a,\ \therefore A=\frac{a}{b};$$

$$Bb-Ac=0;\ \therefore Bb=Ac,\ \therefore B=\frac{Ac}{b}=\frac{ac}{b^2};$$

$$Cb-Bc=0;\ \therefore Cb=Bc,\ \therefore C=\frac{Bc}{b}=\frac{ac^2}{b^3};$$

$$Db-Cc=0;\ \therefore Db=Cc,\ \therefore D=\frac{Cc}{b}=\frac{ac^3}{b^4};$$

$$\text{Therefore } \frac{a}{b-cx}=\frac{a}{b}+\frac{acx}{b^2}+\frac{ac^2x^2}{b^3}+\frac{ac^3x^3}{b^4}\ \&c.\ \text{Ans.}$$

5. Find the expansion of the expression $\sqrt{(a-x)}$.

Assume $\surd(a-x)=A+Bx+Cx^2+Dx^3+Ex^4+\&c.$

Clearing the equation of surds by squaring,

$a-x=A^2+2ABx+(B^2+2AC)x^2+(2AD+2BC)x^3+\&c.$

By transposition, $(A^2-a)+(2AB+1)x+(2AC+B^2)x^2+(2AD+2BC)x^3+\&c.=0$. Therefore $A^2-a=0$,

$$\therefore\ A=a^{\frac{1}{2}};\quad 2AB+1=0,\ \therefore\ 2AB=-1,\ \therefore\ B=-\frac{1}{2A}=-\frac{a^{\frac{1}{2}}}{2a}.$$

$$2AC+B^2=0,\quad \therefore\ 2AC=-B^2,\ \therefore\ C=-\frac{B^2}{2A}=-\frac{a^{\frac{1}{2}}}{8a^3};$$

$$2AD+2BC=0,\quad \therefore\ AD=-BC,\ \therefore\ D=-\frac{BC}{A}=-\frac{a^{\frac{1}{2}}}{16a^5};$$

Whence $\surd(a-x)=a^{\frac{1}{2}}\{1-\frac{x}{2a}-\frac{x^2}{8a^3}-\frac{x^3}{16a^5},\ \&c.\}$ Ans.

6. Required the development of $\surd(a^2+x^2)$ by this method.

Assume $\surd(a^2+x^2)=A+Bx+Cx^2+Dy^3+\&c.$; then, by squaring each side, and transposing, we have

$$\left.\begin{array}{lll}A^2 & +2ABx & +2AC\\ -a^2 & & +B^2\\ & & +1\end{array}\right\}x^2\left.\begin{array}{l}+2AD\\ +2BC\end{array}\right\}x^2\left.\begin{array}{l}+2BD\\ +C^2\end{array}\right\}x^4+\&c.=0;$$

$$\text{Whence}\left\{\begin{array}{ll}A^2-a^2=0, & \text{therefore } A=a\\ 2AB=0, & \dots\dots\dots B=0\\ 2AC-1=0, & \dots\dots\dots C=\frac{1}{2a}\\ 2AD+2BC=0,\ \&c. & \dots\dots\dots D=0,\ \&c.\end{array}\right.$$

Therefore $\surd(a^2+x^2)=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\&c$

7. Required the development of $\frac{x}{1+x+x^2}$ by the same method. Here, since the first term of the series must contain x,

Assume $\frac{x}{1+x+x^2}=Ax+Bx^2+Cx^3+Dx^4+\&c.$ then we have

$$\left.\begin{array}{r}A\\ -1\end{array}\right\}x\left.\begin{array}{l}+A\\ +B\end{array}\right\}x^2\left.\begin{array}{l}+A\\ +B\\ +C\end{array}\right\}x^3\left.\begin{array}{l}+B\\ +C\\ +D\end{array}\right\}x^4\left.\begin{array}{l}+C\\ +D\\ +E\end{array}\right\}x^5+\&c.=0;$$

$$\text{Whence}\left\{\begin{array}{lll}A-1 & =0, & \text{therefore } A=1\\ A+B & =0, & \dots\dots\dots B=-1\\ A+B+C & =0, & \dots\dots\dots C=0\\ B+C+D & =0, & \dots\dots\dots D=1\\ C+D+E & =0, & \dots\dots\dots E=-1;\end{array}\right.$$

Therefore $\frac{x}{1+x+x^2}=x-x^3+x^4-x^5+x^7-\&c.$

1. It is required to convert $\frac{A}{b-ax}$ into a series by the above method. Ans. $\frac{A}{b}(1+\frac{ax}{b}+\frac{a^2x^2}{b^2}+\frac{a^3x^3}{b^3}+\frac{a^4x^4}{b^4}+\&c.$

2. It is required to convert $\frac{1+2x}{1-x-x}$ into a series by the above method. Ans. $a+3x+4x^2+7x^3+11x^4+18x^5+29x^6+\&c.$

3. It is required to convert $\frac{1-x}{1-2x-3x^2}$ into a series by the same method. Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6+\&c.$

Assume $\frac{1-x}{1-2x-3x^2}=A+Bx+Cx^2-Dx^3-Ex^4+Fx^5+Gx^6+,\&c,$

This, cleared of fractions, and by transposition, we have

$$\left.\begin{array}{r|r|r|r|r|r|l} A+\;B & x+\;C & x^2+\;D & x^3+\;E & x^4+\;F & x^5+\;G & x^6+\&c. \\ -1-2A & -2B & -2C & -2D & -2E & -2F & -\&c. \\ +1 & -3A & -3B & -3C & -3D & -3E & \end{array}\right\}=0.$$

Therefore $A-1=0$; $\therefore A=1$.

$B-2A+1=0 \quad B=2A-1=2-1=1;$

$C-2B-3A=0;\ \therefore C=2B+3A=2.2+3=5$

$D-2C-3B=0;\ \therefore D=2C+3B=2.5+3.1=13$

$E-2D-3C=0;\ \therefore E=2D+3C=2.13+3.5=26+15=41$

$F-2E-3D=0;\ \therefore F=2E=3D=2.41+3.13=82+39=121$

$G-2F-3E=0;\ \therefore G=2F=3E=2.121+3.41=242+123=365$

$\frac{1-x}{1-2x-3x^2}=1+x+5x^2+13x^3+41x^4+121x^5+365x^5+\&c.$ Ans.

4. It is required to convert $\sqrt{(1-x)}$ into a series by the same method. Ans. $1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\frac{3.5.7x^5}{2.4.6.8.10}-\&c.$

5. It is required to find, according to the above method, the several roots or values of x, in the equation $x^4-6x^3+13x^2-12x=a$, by means of a quadratic equation.

Ans. $x=\frac{1}{4}\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+)}\}}$.

Put $x=y+z$; then by substitution and ordering the equation, the terms I have $y^4+y^3(4z-6)+y^2(6z^2-18z+13)+y(4z-6)\times(z^2-3z+2)+z^4-6z^3+13z^2-12z=a$. (1.) Let $4z-6=0$; or $z=\frac{3}{2}$; then the odd powers of y disappear from the (1), and it becomes, by substitution for z, its value in the other terms, $y^4-\frac{1}{2}y^2-\frac{63}{16}=1$, or $y^4-\frac{1}{2}y^2=a+\frac{63}{16}$. Hence, by completing the square, I have $y^4-\frac{1}{2}y^2+\frac{1}{16}=a+4\ \therefore y^2=\frac{1}{4}\pm\sqrt{(a+4)}$, or $y=\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+4)}\}}$; hence $x=y+z=\frac{3}{2}\pm\sqrt{\{\frac{1}{4}\pm\sqrt{(a+4)}}$.

6. Convert $\frac{1-x}{1-3x-2x^2}$, or $\frac{1}{1-2x+x^2}$, or $\frac{1+x}{1-2x+x^2}$, or

$\frac{3-x-6x^2}{1-2x-x^2+2x^3}$ and $\frac{1}{1-2ax+x^2}$, or $\frac{1+5x}{1-x-6x^2}$, or $\frac{2}{1+2x}$, or $\frac{1}{(1+x)^2}$, or $\frac{1+x+2x^2}{(1-x)^2-2x^3}$, and $\frac{1+6x+x^2}{(1-x)^4}$, each of the foregoing expressions by the method of indeterminate coefficients.

Answers: $1+2x+8x^2+28x^3+100x^4+356x^5+$&c., or $1+2x+3x^2+5x^3+8x^4+13x^2+$&c., or $1+3x+5x^2+7x^3+$&c. or $3+5x+7x^2+13x^3+23x^4+$&c., or $1+2ax+(4a^2-1)x^2+(8a^3-4a)x^3+$&c., or $1+6x+12x^2+48x^3+120x^5+$&c., or $2-4x+8x^2-16x^3+32x^4$&c., or $1+2x+3x^2+4x^3+5x^4+6x^5+$&c. or $1+3x+7x^2+13x^3+25x^4+51x^5+103x^6+$&c., or $1+8x+27x^2+64x^3+125x^4+216x^5+343x^6$.

8 Assume $\frac{1-x}{1-3x-2x^2}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$

This, cleared of fractions, and by transposition, we have

$$\left.\begin{array}{r|l|l|l|l|l|l|l} A\ +\ B & x+\ C & x^2+\ D & x^3+\ E & x^4+\ F & x^5+\ G & x^6+\ H & x^7 \\ +\ 1 & x-3B & x^2-3C & x^3-3D & x^4-3E & x^5-3F & x^6-3G & x^7- \\ -\ 1 & -\ 2A & x^2-2B & x^3-2C & x^4-2D & x^5-2E & x^6-2F & x^7- \end{array}\right\}=0,$$

$A-1=0$ and $A=1$ $\quad 1$

$B-3A+1=0, \therefore B=3A-1=2,.$ $\quad 2x$

$C-3B-2A=0; \therefore C=3B+2A=8$ $\quad 8x^2$

$D-3C-2B=0; \therefore D=3C+2B=28$ $\quad 28x^3$

$E-3D-2C=0; \therefore E=3D+2C=100$ $\quad 100x^4$

$F-3E-2D=0; : \therefore F=3E+2D=356$ $\quad 356x^5$

$G-3F-2E=0; \therefore G=3F+2E=1268$ $\quad 1268x^6$

$\frac{1-n}{1-3x-2x}=1+2x+8x^2+28x^3+100x^4+356x^5+1268x^6$ Ans

9 Assume $\frac{a}{a'+b'x}=A+Bx+Cx^2+Dx^3+$&c.;

Then multiplying each side by $a'+b'x$, and transposing, we have
$$\left.\begin{array}{r} Aa' \\ -a \end{array}\right\} x \left.\begin{array}{r} +Aa' \\ +Bb' \end{array}\right\} x^2 \left.\begin{array}{r} +Da' \\ +Cb' \end{array}\right\} x^3+\text{&c.}=0;$$

$$\text{Whence}\begin{cases} Aa'-a=0, \text{ therefore } A=\frac{a}{a'} \\ Ba'+Ab'=0 \quad \dots \quad B=-\frac{b'}{a'}A \\ Ca'+Bb'=0 \quad \dots \quad C=-\frac{b'}{a'}B \\ Da'+Cb'=0 \quad \dots \quad D=-\frac{b'}{a'}C\text{&c.} \end{cases}$$

Of the Binomial Theorem.

116. The Binomial Theorem is a general algebraical expression, or formula, by which any power, or root of a given quantity, consisting of two terms, is expanded into a series; the form of which, as it was first proposed by *Newton*, being as follows:

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}[1+\frac{m}{n}Q+\frac{m}{n}(\frac{m-n}{2n})Q^2+\frac{m}{n}(\frac{m-n}{2n})(\frac{m-2n}{3n})Q^3+$$

or

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+$$

&c. where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root, and A, B, C, &c. the terms immediately preceding those in which they are first found, including their signs + or —.

Which theorem may be applied to any particular case, by substituting the numbers or letters in the given example, for P, Q, *m* and *n*, in either of the above formulæ, and then finding the result according to the rule.

When the index of the binomial is a whole number, the series will terminate, as observed under the article Involution before treated on; but when it is a negative or fractional number, as in the following examples, the series will proceed on ad infinitum, and will become more convergent the less the second term of the binomial is with respect to the first.*

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* This celebrated theorem, which is of the most extensive use in Algebra, and various other branches of analysis, may be otherwise expressed as follows:

$$(1)\ (a+x)^{\frac{m}{n}}=a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m-n}{2n}(\frac{x}{a})^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}(\frac{x}{a})^3+\&c.$$

$$(2)\ \text{or}\ (a+x)^{\frac{m}{n}}=$$

$$a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a+x})^2+\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{x}{a+x})^3+$$

$$(3)\ \text{or}\ (a+x)^{\frac{m}{n}}=$$

$$(2a)^{\frac{m}{n}}\{1-\frac{m}{n}(\frac{a-x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{a-x}{a+x})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{a-x}{a+x})^3+$$

And if the reciprocals of the same expressions be required, they will be

$$(4)\frac{1}{(a+x)^{\frac{m}{n}}}=\frac{1}{a^{\frac{m}{n}}}\{1-\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{m+2n}{3n}(\frac{x}{a})^3$$

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Here $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$. and $n=2$; whence

$$P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A,\quad \frac{m}{n}AQ=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x^2}{2.4a^3}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{x}{a^2}=\frac{3x^3}{2.4.6a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=E,$$

$$\frac{m-4n}{5n}EQ=\frac{1-8}{10}\times-\frac{3.5.x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=F,\ \ldots$$

$$(a^2+x)^{\frac{1}{2}}=a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}$$

Where the law of formation of the several terms of the series is sufficiently evident.

2. It is required to convert $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$, into an infinite series.

Here $P=a$, $Q=\frac{b}{a}$, and $\frac{m}{n}=-2$, or $m=-2$ and $n=1$;

$$(5)\text{ or }\frac{1}{(a+x)^{\frac{m}{n}}}=\left(\frac{1}{a}\right)^{\frac{m}{n}}\left\{1-\frac{m}{n}\left(\frac{x}{a+x}\right)+\frac{m}{n}.\frac{m-n}{2n}\left(\frac{x}{a+x}\right)^2-\right.$$

$$\left.\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\left(\frac{x}{a+x}\right)^3+\&c.\right\};\quad (6)\ \frac{1}{(a+x)^{\frac{m}{n}}}=\left(\frac{1}{2a}\right)^{\frac{m}{n}}\left\{1+\frac{m}{n}\left(\frac{a-x}{a+x}\right)\right.$$

$$\left.+\frac{m}{n}.\frac{m+n}{2n}\left(\frac{a-x}{a+x}\right)^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\left(\frac{a-x}{a+x}\right)^3+\&c.\right\}$$

It may here also be observed, that if m, taken singly, be made to represent any whole, or fractional number, whether positive or negative, the first of these expressions may be exhibited under the more simple form

$$(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^{m-3}x^3.$$

$$+\ \ldots\ldots\ldots+\frac{m(m-1)(m-2)\ldots\ldots\ldots\{m-(n-1)\}a^{m-n}x^n}{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots\ldots n}$$

where the last term is called the general term of the series, because if 1, 2, 3, 4, &c. be substituted successively for n, it will give all the rest.

whence $P^{\frac{m}{n}}=(a)^{\frac{m}{n}}=a^{-2}=\frac{1}{a^2}=A$, $\frac{m}{n}AQ=-\frac{2}{1}\times\frac{1}{a^2}\times\frac{b}{a}=-\frac{2b}{a^3}=B$; $\frac{m-n}{2n}BQ=\frac{-2-1}{2}\times-\frac{2b}{a^3}\times\frac{b}{a}=\frac{3b^2}{a^4}=C$,

$$\frac{m-2n}{3n}CQ=\frac{-2-2}{3}\times\frac{3b^2}{a^4}\times\frac{b}{a}=-\frac{4b^3}{a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-2-3}{4}\times-\frac{4b^3}{a^5}\times\frac{b}{a}=\frac{5b^4}{a^6}=E \text{ \&c. \&c.}$$

Consequently $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^2}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-$&c.

3. It is required to convert $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}$, or its equal $a^2(a^2-x)^{-\frac{1}{2}}$ into an infinite series. Here $P=a^2$, $Q=-\frac{x}{a^2}$, and $\frac{m}{n}=\frac{-1}{2}$, or $m=-1$ and $n=2$; whence $P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=\frac{1}{a}=A$, $\frac{m}{n}AQ=-\frac{1}{2}\times\frac{1}{a}\times-\frac{x}{a^2}=\frac{x}{2a^3}=B$,

$$\frac{m-n}{3n}BQ=\frac{-1-2}{4}\times\frac{x}{2a^3}\times-\frac{x}{a^2}=\frac{3x^2}{2.4a^5}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-1-4}{6}\times\frac{3x^2}{2.4a^5}\times-\frac{x}{a^2}=\frac{3.5x^3}{2.4.6a^7}=D,$$

$\frac{m-3n}{4n}DQ=\frac{-1-6}{8}\times\frac{3.5x^3}{2.4.6a^7}\times-\frac{x}{a^2}=\frac{3.5.7x^4}{2.4.6.8a^9}=E$,&c; therefore $\frac{1}{(a^2-x)^{\frac{1}{2}}}=\frac{1}{2}a+\frac{1}{2}\{\frac{x}{a^3}\}+\frac{3}{2.4}\{\frac{x^2}{a^5}\}+\frac{3.5}{2.4.6}\{\frac{x^3}{a^7}\}+\frac{3.5.7}{2.4.6.8}\{\frac{x^4}{a^9}\}$ +&c; $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}=a+\frac{1}{2}\{\frac{x}{a}\}+\frac{3}{2.4}\{\frac{x^2}{a^2}\}+\frac{3.5}{2.4.6}\{\frac{x^3}{a^5}\}+\frac{3.5.7}{2.4.6.8}\{\frac{x^4}{a^7}\}$.

4. It is required to convert $\sqrt[3]{9}$, or its equal $(8+1)^{\frac{1}{3}}$, into an infinite series.

Here $P=8$, $Q=\frac{1}{8}$, and $\frac{m}{n}=\frac{1}{3}$, or $m=1$ and $n=3$; whence $P^{\frac{m}{n}}=(8)^{\frac{m}{n}}=8^{\frac{1}{3}}=2=A$, $\frac{m}{n}AQ=\frac{1}{3}\times\frac{2}{1}\times\frac{1}{2^3}=\frac{1}{3.2^2}=B$,

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=D,$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.8.12.2^{10}}=\text{E},$$

$$\frac{m-4n}{5n}\text{EQ}=\frac{1-12}{15}\times-\frac{5.8}{3.6.9.12.2^{10}}\times\frac{1}{2^3}=\frac{5.8.11}{3.6.9.12.15.2^{13}}=\text{F},$$

$$\sqrt[3]{9}=2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\frac{5.8.11}{3.6.9.12.15.2^{13}}$$

—&c., which series, in this instance, converges very rapidly; and the same will be the case with the example fifth next following. But when the given number falls in, or about the middle, between the two nearest integral powers, this will not take place.

In such cases, therefore, recourse must be had to the use of logarithms, the rule of double position, or some other more ready mode of computation.

5. Required to convert $\sqrt[5]{243}$, or $\sqrt{(1+1)}$, $(a\pm x)^{\frac{1}{2}}$, $(a\pm b)^{\frac{1}{3}}$, $(a-b)^{\frac{1}{4}}$, $(a+x)^{\frac{2}{3}}$, and $(1-x)^{\frac{2}{5}}$, each expression to an infinite series.

6. Convert $\frac{1}{(a\pm x)^{\frac{1}{2}}}$, or $\frac{a}{(a\pm x)^{\frac{1}{3}}}$, or $\frac{1}{(1+x)^{\frac{1}{5}}}$, or $\left(\frac{1+x}{a-x}\right)^{\frac{1}{2}}$, or their =s, $(a\pm x)^{-\frac{1}{2}}$, $a(a\pm x)^{-\frac{1}{3}}$, $(1+x)^{-\frac{1}{5}}$, and $(1+x)^{\frac{1}{2}}(a-x)^{-\frac{1}{2}}$ each into an infinite series.

5,. Here P=243, $\text{A}=-\frac{3}{243}=\frac{-1}{3^4}$, $\frac{m}{n}=\frac{1}{5}$, or $m=1$, and $n=5$;

$$\text{P}^{\frac{m}{n}}=243^{\frac{1}{5}}=3=\text{A},\ \frac{m}{n}\text{AQ}=\tfrac{1}{5}\times3\times-\frac{1}{3^4}=\frac{-1}{5.3^3}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-4}{10}\times\frac{-1}{5.3^3}\times-\frac{1}{3^4}=\frac{-4}{5.10.3^7}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{-9}{15}\times\frac{-4}{5.10.3^7}\times\frac{-1}{3^4}=\frac{-4.9}{5.10.15.3^{11}}=\text{D};$$

$$\frac{m-3n}{4n}\text{DQ}=-\frac{14}{20}\times\frac{-4.9}{5.10.15.3^{11}}\times\frac{-1}{3^4}=\frac{-4.9.14}{5.10.15.20.3^{15}}=\text{E};$$

$$\sqrt[5]{240}=3-\frac{1}{5\cdot3^3}-\frac{4}{5\cdot10.3^7}-\frac{4.9}{5.10.15.3^{11}}-\frac{4.9.14}{5.10.15.20.3^{15}}-\&c.$$

5,'. Here A=a, $q=\pm\frac{x}{a}$, $m=1$, $n=2$, $\text{P}^{\frac{m}{n}}=a^{\frac{1}{2}}=\text{A}$,

$$\frac{m}{n}\text{AQ}=\tfrac{1}{2}\times a^{\frac{1}{2}}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times\pm\frac{x}{2a}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{-1}{4}\times a^{\frac{1}{2}}\times\pm\frac{x}{2a}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times-\frac{x^2}{2.4.a^2}=\text{C},$$

$$\frac{m-2n}{3n}CQ=\frac{-3}{6}\times a^{\frac{1}{2}}\times-\frac{x^2}{2.4a^2}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times\pm\frac{3x^3}{2.4.6a^3}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{5}{8}\times a^{\frac{1}{2}}\times\pm\frac{3x^3}{2.4.6a^3}\times\pm\frac{x}{a}=a^{\frac{1}{2}}\times-\frac{3.5.x^4}{2.4.68a^4.}=E$$

$$\therefore\ \surd(a\pm x)=a^{\frac{1}{2}}\times(1\pm\frac{x}{2a}-\frac{x^2}{2.4a^2}\pm\frac{3x^3}{2.4.6a^3}-\frac{3.5.x^4}{2.4.6.8a^4}\pm\&c.$$

'5,'. Here $P=a$, $q=\pm\frac{b}{a}$, $m=1$, and $n=3$. $P^{\frac{1}{3}}=a^{\frac{1}{3}}=A$,

$$\frac{m}{n}AQ=\tfrac{1}{3}a^{\frac{1}{3}}\times\pm\frac{b}{a}=a^{\frac{1}{3}}\times\pm\frac{b}{3.a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-2}{6}\times a^{\frac{1}{3}}\times\pm\frac{b}{3.a}\times\pm\frac{b}{a}=a^{\frac{1}{3}}\times\frac{-2b^2}{3.6a^2}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-5}{9}\times a^{\frac{1}{3}}\times\frac{-2b^2}{3.6.a^2}\times\pm\frac{b}{a}=a^{\frac{1}{3}}\times\pm\frac{2.5.b^3}{3.6.9a^3}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-8}{12}\times a^{\frac{1}{3}}\times\frac{2.5.b^3}{3.6.9.a^3}\times\pm\frac{b}{a}=\frac{a^{\frac{1}{3}}\times-2.5.8b^4}{3.6.9.12a^4}=E;$$

$$\therefore\ (a\pm b)^{\frac{1}{3}}=a^{\frac{1}{3}}\times(1\pm\frac{b}{3a}-\frac{2b^2}{3.6a^2}\pm\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}\pm\&c.$$

',6,'. Here $P=a$, $q=-\frac{b}{a}$, $m=1$, $n=4$. $P^{\frac{m}{n}}=a^{\frac{1}{4}}=A$,

$$\frac{m}{m}AQ=\tfrac{1}{4}\times a^{\frac{1}{4}}\times-\frac{b}{a}=a^{\frac{1}{4}}\times-\frac{b}{4a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-3}{8}\times a^{\frac{1}{4}}\times-\frac{b}{4a}\times-\frac{b}{a}=a^{\frac{1}{4}}\times\frac{-3b^2}{4.8a^2}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-7}{12}\times a^{\frac{1}{4}}\times\frac{-3b^2}{4.8a^2}\times-\frac{b}{a}=a^{\frac{1}{4}}\times-\frac{3.7b^3}{4.8.12a^3}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-11}{16}\times a^{\frac{1}{4}}\times-\frac{3.7b^3}{4.8.12a^3}\times-\frac{b}{a}=\frac{a^{\frac{1}{4}}\times-3.7.11b^4}{4.8.12.16a^4}=$$

$$\therefore\ (a-b)^{\frac{1}{4}}=a^{\frac{1}{4}}\times(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\&c$$

',,5,'. Here $P=a$, $Q=\frac{x}{a}$, $m=2$, $n=3$. $P^{\frac{m}{n}}=a^{\frac{2}{3}}=A$,

$$\frac{m}{n}AQ=\tfrac{2}{3}a^{\frac{2}{3}}\times\frac{x}{a}=a^{\frac{2}{3}}\times\frac{2x}{3a}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-1}{6}\times a^{\frac{2}{3}}\times\frac{2x}{3x}\times\frac{x}{a}=a^{\frac{2}{3}}\times-\frac{x^2}{9a^2}=C,$$

$$\frac{m-2n}{3n}CQ=-\tfrac{4}{9}\times a^{\frac{2}{3}}\times\frac{-x^2}{9a^2}\times\frac{x}{a}=a^{\frac{2}{3}}\times\frac{4x^3}{9^2a^3}=D,$$

$$\frac{m-3n}{4n}\mathrm{DQ}=-\frac{7}{12}\times a^{\frac{2}{3}}\times\frac{4x^3}{9^2a^3}\times\frac{x}{a}=a^{\frac{2}{3}}\times\frac{-4.7x^4}{9^2.12a^4}=\mathrm{E},$$

$$\therefore(a+x)^{\frac{2}{3}}=a^{\frac{2}{3}}\left\{1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4.7x^4}{9^2.12a^4}+\&c.\right.$$

6′. Here $\mathrm{P}=1$, $\mathrm{Q}=-x$, $m=2$, $n=5$.

$$\mathrm{P}^{\frac{m}{n}}=1=\mathrm{A},\ \frac{m}{n}\mathrm{AQ}=\tfrac{2}{5}\times-x=\frac{-2x}{5}=\mathrm{B},$$

$$\frac{m-n}{2n}\mathrm{BQ}=\frac{-3}{10}\times-\frac{2x}{5}\times-x=\frac{-2.3x^2}{5.10}=\mathrm{C},$$

$$\frac{m-2n}{3n}\mathrm{CQ}=\frac{-8}{15}\times-\frac{2.3x^2}{5.10}x=\frac{-2.3.8x^3}{5.10.15}=\mathrm{D},$$

$$\frac{n-3n}{4n}\mathrm{DQ}=\frac{-13}{20}\times-\frac{2.3.8x^3}{5.10.15}\times-x=\frac{-2.3.8.13x^4}{5.10.15.20}=\mathrm{E},$$

$$\therefore(1-a)^{\frac{2}{5}}=1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}-\&c.$$

6″. Here $\mathrm{P}=a$, $\mathrm{Q}=\pm\frac{x}{a}$, $m=-1$, $n=2$. $\mathrm{P}^{\frac{m}{n}}=a^{-\frac{1}{2}}=\mathrm{A}$

$$\frac{m}{n}\mathrm{AQ}=\frac{-1}{2}a^{-\frac{1}{2}}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\mp\frac{x}{2.a}=\mathrm{B},$$

$$\frac{m-n}{2n}\mathrm{BQ}=\frac{-3}{4}\times a^{-\frac{1}{2}}\times\mp\frac{x}{2a}\times\pm\frac{x^2}{a}=a^{-\frac{1}{2}}\times\frac{3x^2}{2.4.a^2}=\mathrm{C},$$

$$\frac{m-2n}{3n}\mathrm{CQ}=\frac{-5}{6}\times a^{-\frac{1}{2}}\times\frac{3x^2}{2.4.a^2}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\mp\frac{3.5x^3}{2.4.6a^2}=$$

$$\frac{m-3n}{4n}\mathrm{DQ}=\frac{-7}{8}\times a^{-\frac{1}{2}}\mp\frac{3.5x^3}{2.4.6a^3}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\frac{3.5.7x^4}{2.4.6.8a^4}=\mathrm{E};$$

$$\therefore(a\pm x)^{-\frac{1}{2}}=a^{-\frac{1}{2}}(1\mp\frac{x}{2.a}+\frac{3x^2}{2.4a^2}\mp\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4}\mp\&c.$$

6‴. Here $\mathrm{P}=a$, $\mathrm{Q}=\pm\frac{x}{a}$, $m=-1$, $n=3$. $\mathrm{P}^{\frac{m}{n}}=a^{-\frac{1}{3}}=\mathrm{A}$,

$$\frac{m}{n}\mathrm{AQ}=\frac{-1}{3}a^{-\frac{1}{3}}\times\pm\frac{x}{a}=a^{-\frac{1}{3}}\times\mp\frac{x}{3.a}=\mathrm{B},$$

$$\frac{m-n}{2n}\mathrm{BQ}=\frac{-4}{6}\times a^{-\frac{1}{3}}\times\pm\frac{x}{3a}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\frac{4x^2}{3.6a^2}=\mathrm{C},$$

$$\frac{m-n}{3n}\mathrm{CQ}=\frac{-7}{9}\times a^{-\frac{1}{3}}\times\frac{4x^2}{3.6a^2}\times\pm\frac{x}{a}=a^{-\frac{1}{3}}\times\mp\frac{4.7x^3}{3.6.9a^3}=\mathrm{D}$$

$$\frac{m-3n}{4n}\mathrm{DQ}=\frac{10}{12}\times a^{-\frac{1}{3}}\mp\frac{4.7x^3}{3.6.9a^3}\times\pm\frac{x}{a}=a^{-\frac{1}{2}}\times\frac{4.7.10x^4}{3.6.9.12a^4}=\mathrm{E};$$

$$\therefore(a\pm x)^{-\frac{1}{3}}=a^{-\frac{1}{3}}\times(1\mp\frac{x}{3.a}+\frac{4x^2}{3.6a^2}\mp\frac{4.7x^3}{3.6.9a^3}+\frac{4.7.10x^4}{3.6.9.12a^4}\pm\&c$$

6⁗. Here $\mathrm{P}=1$, $\mathrm{Q}=x$, $m=-1$, $n=5$. See Index for last.

$$P^{\frac{m}{n}}=1=A,\quad \frac{m}{n}AQ=-\frac{1}{5}\times x=-\frac{x}{5}=B,$$

$$\frac{m-n}{2n}BQ=-\frac{6}{10}\times-\frac{x}{5}\times x=\frac{6x^2}{5.10}=C,$$

$$\frac{m-2n}{3n}CQ=-\frac{11}{15}\times\frac{6x^2}{5.10}\times x=\frac{-6.11x^3}{5.10.15}=D,$$

$$\frac{m-3n}{4n}DQ=-\frac{16}{20}\times\frac{-6.11x^3}{5.10.15}\times x=\frac{6.11.16x^4}{5.10.15.20}=E;$$

$$\therefore (1+x)^{-\frac{1}{5}}=1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-\&c.$$

We have seen that $(a\pm x)^2=a^2\pm 2ax+x^2$; $(a\pm x)^3=a^3\pm 3a^2x+3ax^2\pm x^3$; $(a\pm x)^4=a^4\pm 4a^3x+6a^2x^2\pm 4ax^3+x^4$, and in general, $(a\pm x)^m=a^m\pm ma^{m-1}x+\frac{m\cdot(m-1)}{1.2}a^{m-2}x^2\pm\frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}x^3+\&c.$, where m is any number whatever, whole or fractional, positive or negative.

This theorem will, therefore, not only enable us to find any given power of a binomial, but will furnish a ready way of extracting any proposed root of the same. Any binomial, as $(a+x)^m$ may be expanded into a series of the form $A+Bx+Cx^2+\&c.$ whatever be the value of m, where the first term is independent of x, and the following terms are whole positive increasing powers of x, with coefficients B, C, &c. independent of x.

Let m be any whole number, and let $(1+x)^{m-1}=1+Px+Qx^2+\&c.$; $\therefore$ $(1+x)^m=(1+Px+Qx^2+\&c.)\cdot(1+x)$.

$$\left.\begin{array}{l}=1+Px+Qx^2+\&c.\\ \quad+x+Px^2+\&c.\end{array}\right\}=1+(P+1)\cdot x+(P+Q)\cdot x^2+\&c.$$

which shews that, by the introduction of another factor, the coefficient of the second term is increased by 1. Now $(1+x)^2=1+2x+\&c$; $(1+x)^3=1+3x+$ $\therefore$ generally $(1+x)^m=1+mx+$ &c. & $(1+x)^{-m}$, or by actual division. $\frac{1}{(1+x)^m}=1-mx+\&c.$ Now since

$$a+x=a\left(1+\frac{x}{a}\right),\ \therefore (a+x)^m=a^m\left(1+\frac{x}{a}\right)^m=a^m\left(1+m\frac{x}{a}+\&c\right)$$

Hence, whatever be the value of m, the two first terms of the expansion $(a+x)^m$ are $a^m+ma^{m-1}x$. Assume $(a+x)^m=a^m+ma^{m-1}x+Ax^2+Bx^3+$ &c., where A, B, &c. do not involve x, and remain to be determined. On this supposition

$$\{a+(x+z)\}=a^m+ma^{m-1}(x+z)+A(x+z)^2+B(x+z)^3+\&c.$$

$$\left.\begin{array}{l}=a^m+ma^{m-1}x+Ax^2+Bx^3+\&c.\\ \quad+ma^{m-1}z+2Axz+3Bx^2z+\&c.\\ \quad\quad+\&c.\end{array}\right\}\ldots\ldots\ldots p$$

Now $\{(a+x\}+z) = (a+x)^m + m(a+x)^{m-1}z + Az^2 + Bz^3 + \&c. =$

$$\left.\begin{array}{l} = a^m + ma^{m-1}x + Ax^2 + Bx^3 + \&c. \\ \quad + mz\{a^{m-1} + (m-1)\cdot a^{m-2}x + A^1x^2 + B^1x^3 + \&c.\} \\ \qquad\qquad + \&c. \end{array}\right\} \ldots\ldots q,$$

Where a^1, B^1, &c. are the values which A, B, &c. assume when $(m-1)$ is put for m. Now these two series p, q, expressing the values of $(a+x+z)^m$ must be identical; therefore, by equating coefficients $2A = m\cdot(m-1)^{m-2}$, $\therefore A = \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}$; also $A^1 = \frac{(m-1)\cdot(m-2)}{1\cdot 2}a^{m-3}$. Hence $2B = mA^1 = \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2}a^{m-3}$. $\therefore B = \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}$; &c. $\therefore (a+x)^m = a^m + ma^{m-1}x + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 + \frac{m\cdot(m-1)(m-2)}{1\cdot 2.3}a^{m-3}x^3 + \&c.$

In the same manner it will appear that $(a-x)^m = a^m - ma^{m-1}x + \frac{m\cdot(m-1)}{1\cdot 2}a^{m\,|\,2}x^2 - \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}x^3 + \&c. \ldots \pm x^m$.

And this formula will hold good whatever be the value of m; whether it be integral or fractional, positive or negative.

Extract the cube root of 750, or of 9^3+21. Here $e=9$, $m=3$, and $x=21$ $\therefore \sqrt[3]{(750)} = 9\left\{1 + \frac{21}{2250} + \frac{4(21)^2}{18(750)^2} + \&c.\right\} =$ 9·085592, .

If e^m be large compared with x, these expressions converge with great rapidity; and in general it is not necessary to take more than four terms of the series.

Since $(a+x)^m = a^m + ma^{m-1}x + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2$ &c., and $(a-x)^m = a^m - ma^{m-1}x + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 - \&c.$ $\therefore$ by addition, $\frac{(a+x)^m + (a-x)^m}{2} = a^m + \frac{m\cdot(m-1)}{1\cdot 2}a^{m-2}x^2 + \&c.$ = sum of the odd terms of the expansion of $(a+x)^m$. By subtraction $\frac{(a+x)^m - (a-x)^m}{2} = ma^{m-1}x + \frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}a^{m-3}x^3 + \&c.$ = sum of the even terms. If $a=x=1$, $(1+1)^m$ or $2^m = 1 + m + \frac{m\cdot(m-1)}{2} + \&c.$ = sum of the coefficients of the expansion $(a+x)^m$. The sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms: For if $a=x=1$,

$$(a-x)^m = 0;\ \therefore \frac{(a+x)^m + (a-x)^m}{2} = \frac{(a+x)^m - (a-x)^m}{2} = 2^{m-1}.$$

A trinomial $(a+b+x)$ may be raised to any power, thus: $a+b+x=a+(b+x)$; $\therefore\ (a+b+x)^m=(a+(b+x))^m=a^m+ma^{m-1}(b+x)+\frac{m\cdot(m-\mathrm{I})}{1\cdot2}a^{m-2}(b+x)^2+\frac{m\cdot(m-1)\cdot(m-2)}{1\cdot2\cdot3}a^{m-3}(b+x)^3+\text{\&c.}$ by considering $(b+x)$ as one quantity. And in general

$$(a+b+c+d+\text{\&c.})^m=a^m+ma^{m-1}b+ma^{m-1}c+ma^{m-1}d+\text{\&c.}$$

$+\frac{m\cdot(m-1)}{1\cdot2}a^{m-2}b^2+\frac{m\cdot(m-1)}{1\cdot2}a^{m-2}bc+\text{\&c.}+m\cdot\frac{(m-1)\cdot(m-2)}{1\cdot2\cdot3}a^{m-3}b^3+$ &c., which is De Moivre's Theorem.

On the Multiplication and Division of Series.

117. Any two series may be multiplied or divided by each other, by substituting the numbers or letters of the given example, in the place of their corresponding terms, in the following general formulæ, and then finding the result accordingly.*

$$\begin{array}{l}
A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.}\\
B_0+B_1x+B_2x^2+B_3x^3+B_4x^4+\text{\&c.}
\end{array}$$

$$\text{Product}=\begin{array}{|l|l|l|l|l}
A_0B_0+A_1B_0 & x+A_2B_0 & x^2+A_3B_0 & x^3+A_4B_0 & x^4+\text{\&c.}\\
+A_0B_1 & +A_1B_1 & +A_2B_1 & +A_3B_1 & \\
 & +A_0B_2 & +A_1B_2 & +A_2B_2 & \\
 & & +A_0B_3 & +A_1B_3 & \\
 & & & +A_0B_4 &
\end{array}$$

And if $B_0=A_0$, $B_1=A_1$, $B_2=A_2$, &c. the formula for the square of the upper of the above series, or $(A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.})^2$ will evidently be in that case as follows:

$$\text{Square}=\begin{array}{|l|l|l|l|l}
A_0^2+A_1A_0 & x+A_2A_0 & x^2+A_3A_0 & x^3+A_4A_0 & x^4+\text{\&c.}\\
+A_0A_1 & +A_1A_1 & +A_2A_1 & +A_3A_1 & \\
 & +A_0A_2 & +A_1A_2 & +A_2A_2 & \\
 & & +A_0A_3 & +A_1A_3 & \\
 & & & +A_0A_4 &
\end{array}$$

$$\text{Also, }\frac{A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.}}{1+B_1x+B_2x^2+B_3x^3+B_4x^4+\text{\&c.}}$$

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* In the mode of notation here followed, it is to be observed, that A_0, A_1, A_2, &c. are the coefficients of the several powers of x, so taken, that the inferior figures 0, 1, 2, 3, &c. by which they are denoted shall always agree with the indices of the unknown quantities that are attached to them; by which means the law of the continuation of the terms is, in all cases, rendered obvious, and a conformity between the coefficients and powers of x constantly preserved.

$$\text{Quotient} = \left|\begin{array}{l|l|ll|l} A_0+A_1 & x+A_2 & x^2+A_3 & x^3+A_4 & x^4+\&c. \\ \quad -B_1C_0 & -B_2C_0 & -B_3C_0 & -B_4C_0 & \\ & -B_1C_1 & -B_2C_1 & -B_3C_1 & \\ & & -B_1C_2 & -B_2C_2 & \\ & & & -B_1C_3 & \end{array}\right.$$

And, if A_0, in the upper series, be taken $= 1$, and the rest of the terms be put $= 0$, or made to vanish, we shall have for the reciprocal of the lower series, $\dfrac{1}{1+B_1x+B_2x^2+B_3x^3+B_4x^4+\&c.}$

$$\text{Reciprocal} = \left|\begin{array}{l|l|l|l|l} 1-B_1C_0 & x-B_2C_0 & x^2-B_3C_0 & x^3-B_4C_0 & x^4-\&c. \\ & \quad -B_1C_1 & -B_2C_1 & -B_3C_1 & \\ & & -B_1C_2 & -B_2C_2 & \\ & & & -B_1C_3 & \end{array}\right.$$

where it is to be observed, that in each of the two latter series C_0 denotes the first term, C_1 the coefficient of the second term, C_2 that of the third, and so on, according to the order C_3, C_4, &c.*

1. Find the product of the two series $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+$ &c. and $1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\frac{1}{9}x^4-\frac{1}{11}x^5+$ &c.

Here $A_0=1$, $A_1=\frac{1}{2}$, $A_2=\frac{1}{3}$, &c. and $B_0=1$, $B_1=-\frac{1}{3}$, $B_2=\frac{1}{5}$, &c.

Whence $A_0B_0=1$; $A_1B_0+A_0B_1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$;

$A_2B_0+A_1B_1+A_0B_2=\frac{1}{3}-\frac{1}{6}+\frac{1}{5}=\frac{11}{30}$

$A_3B_0+A_2B_1+A_1B_2+A_0B_3=\frac{1}{4}-\frac{1}{9}+\frac{1}{10}-\frac{1}{7}=\frac{121}{1260}$

$A_4B_0+A_3B_1+A_2B_2+A_1B_3+A_0B_4=\frac{1}{5}-\frac{1}{12}+\frac{1}{15}-\frac{1}{14}+\frac{1}{9}=\frac{281}{1260}$, &,

$\therefore 1+\frac{1}{3}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\frac{281}{1260}x^4+$ &c., the product required.

2. Divide the series $1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\frac{281}{1260}x^4+$ &c. by the series $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+$ &c.

Here $A_0=1$, $A_1=\frac{1}{6}$, $A_2=\frac{11}{30}$, &c.; and $B_1=\frac{1}{2}$. $B_2=\frac{1}{3}$, $B_3=\frac{1}{4}$, &c.

Whence $A_0=1=C_0$; $A_1-B_1C_0=\frac{1}{6}-\frac{1}{2}=-\frac{1}{3}=C_1$;

$A_2-B_2C_0-B_1C_1=\frac{11}{30}-\frac{1}{3}+\frac{1}{6}=\frac{1}{5}=C_2$;

$A_3-B_3C_0-B_2C_1-B_1C_2=\frac{121}{1260}-\frac{1}{4}+\frac{1}{9}-\frac{1}{10}=-\frac{1}{7}=C_3$,

$A_4-B_4C_0-B_3C_1-B_2C_2-B_1C_3=\frac{281}{1260}-\frac{1}{5}+\frac{1}{12}-\frac{1}{15}+\frac{1}{14}=\frac{1}{9}=C_4$;

&c. Therefore $1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\frac{1}{9}x^4-$ &c. $=$ quotient.

1. Multiply $1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4+\frac{1}{32}x^5+$ &c. by $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+$ &c. Ans. $1+x+\frac{5}{6}x^2+\frac{2}{3}x^3+\frac{8}{15}x^4+\frac{13}{30}x^5+$ &c.

2. Find the square of the series $1-\frac{1}{2}x+\frac{2}{3}x^2-\frac{3}{4}x^3+\frac{4}{5}x^4-\frac{5}{6}x^5+$

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* The first term of the denominators of the two last examples given above are, for the sake of simplicity, taken equal to unity; to which form any series can always be reduced; as is evident from the following instance,

$$a+bx+cx^2+dx^3+ex^4+\&c. = a(1+\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3+\frac{e}{a}x^4+)\&c.$$

which method may also be used in other cases, whenever it will render a formula more perspicuous, or any process of investigation less embarrassing.

$\frac{6}{7}x^6$—&c. Ans. $1-x+\frac{13}{12}x^2-\frac{13}{6}x^3+\frac{503}{180}x^4-\frac{52}{15}x^5+$&c.

3. Divide $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+$&c. by $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-$&c. Ans. $1+x+\frac{1}{2}x^2+\frac{5}{12}x^3+\frac{7}{24}x^4+\frac{191}{720}x^5+$&c.

4. Find the reciprocal of $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+\frac{1}{6}x^5+$&c.

Ans. $1-\frac{1}{2}x-\frac{1}{12}x^2-\frac{1}{24}x^3-\frac{19}{720}x^3-\frac{27}{1440}x^5-$&c.

On the Multinomial Theorem.

118. The multinomial theorem is a general expression, or formula, for determining any power or root of a given quantity, consisting of any number of terms, the form of which is as follows:

$$(A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots+A_nx^n)^m=A_0{}^m+$$

$$\begin{array}{r}mA_1B_0\\+(m-1)A_1B_1\end{array}\left|\frac{x}{A_0}\right.\;\begin{array}{r}+2mA_2B_0\\+(2m-1)A_2B_1\\+(\ m-2)A_1B_2\end{array}\left|\frac{x^2}{2A_0}\right.\;\begin{array}{r}+3mA_3B_0\\+(3m-1)A_3B_1\\+(2m-2)A_2B_2\\+(\ m-3)A_1B_3\end{array}\left|\frac{x^3}{3A_0}\right.\;+4mA_4B_0\left|\frac{x^4}{4A},\right.\ \&c$$

where $B_0=A_0{}^m$; also B_1, B_2, B_2, &c. are the coefficients of the terms immediately preceding those in which they first appear.

Or the same series may be exhibited in a more convenient and practical form, as follows;

$$(1+Ax+Bx^2+Cx^3+Dx^4+Ex^5+\&c.)^m=1$$

$$+\frac{mA}{1}x+\frac{2mB+(m-1)Aa}{2}x^2+\frac{3mC+(2m-1)Ba+(m-2)Ab}{3}x^3$$

$$+\frac{4mD+(3m-1)Ca+(2m-2)Bb+(m-3)Ac}{4}x^4$$

$$\frac{5mE+(4m-1)Da+(3m-2)Cb+(2m-3)Bc+(m-4)Ad}{5}x^5$$

&c. where *a*, *b*, *c*, &c. are the coefficients of the terms immediately preceding those in which they are first used.

And the manner of applying the theorem to any particular case is by substituting the numbers, or letters, in the given examples, for A_0, A_1, A_2, &c. and *m*, as in the binomial theorem before given.

1. Find the cube of the series $1+x+x^2+x^3+x^4+x^5+$&c.

Here $A_0=1$, $A_1=1$, $A_2=1$, &c. and $m=3$;
whence $A_0{}^m=1^m=1=B_0$; $mA_1B_0=3\times1\times1=3=B_1$;

$$\frac{2mA_2B_0+(m-1)A_1B_1}{2A_0}=\frac{6+2\times3}{2}=6=B_2.$$

$$\frac{3mA_3B_0+(2m-1)A_2B_1+(m-2)A_1B_2}{3A_0}=\frac{9+15+6}{3}=10=B_3;$$

$$\frac{4mA_4B_0+(3m-1)A_3B_1+(2m-2)A_2B_2+(m-3)A_1B_3}{4A_7}=15-B_4\ \&c.$$

Therefore $1+3x+6x^2+10x^3+15x^4+21x^5+$&c. = required cube.

2. Find the cube root of the series $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\frac{1}{5}x^4+\frac{1}{6}x^5+$&c. Here $A_0=1, A_1=\frac{1}{2}, A_2=\frac{1}{3}, A_3=\frac{1}{4}$, &c.; and $m=\frac{1}{3}$, $\therefore\ A_0^m=1^m=1=B_0;\ mA_1B_0=\frac{1}{3}\times\frac{1}{2}\times\frac{1}{1}=\frac{1}{6}=B_1;$

$$\frac{2mA_2B_0+(m-1)A_1B_1}{2A_0}=\tfrac{1}{2}(\tfrac{2}{9}-\tfrac{1}{18})=\tfrac{1}{12}=B_2$$

$$\frac{3mA_3B_0+(2m-1)A_2B_1+(m-2)A_1B_2}{3A_0}=\tfrac{1}{3}(\tfrac{1}{4}-\tfrac{1}{54}-\tfrac{1}{72})=\tfrac{35}{648}=B_3,$$

$$\frac{4mA_4B_0+(2m-1)A_2B_1+(2m-2)A_2B_2+(m-3)A_1B_3}{4A_0}=\tfrac{433}{1944}=B_4,$$

&c.; therefore $1+\frac{1}{6}x-\frac{1}{12}x^2+\frac{35}{648}y^3-\frac{433}{1944}x^4+$&c. Ans.

3. What is the square root of the series $1+x+x^2+x^3+$&c? Here a, b, c, &c. are each $=1$; also $Q=\frac{1}{2}$; therefore

$$a^n=1=A,\ Qb=\tfrac{1}{2}=B,\ \frac{(n-1)Ab}{2a}+Qc=\tfrac{3}{8}=C,$$

$$\frac{(n-2)Bb+(2n-1)}{3a}+Qd=\tfrac{5}{10}=D,$$

$$\frac{(n-3)Cb+(2n-2)Bc+(3n-1)Ad}{4a}+Qe=\tfrac{35}{128}=E;\ \text{\&c.}$$

$\therefore\ (1+x+x^2+x^3+\text{\&c.})^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{124}x^4+$&c.

4. Find the square of the series $1-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-$&c.

5. Find the square of the series $1+x+x^2+x^3+x^4+$&c.

Ans. $1-\frac{2}{3}x^3+\frac{23}{45}x^5-\frac{44}{105}x^7+\frac{121}{315}x^9$&c. Ans. $1+2x+3x^2+4x^3+5x^4$

6. Find the square root of the series $1+x+x^2+x^3+x^4+x^5+$&c.

7. Find the square of the series $y-y^3+y^5-y^7+y^9-$&c.

Ans. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4$. Ans. $y^2-2y^4+3x^6-4y^8+5y^{10}$.

8. Find the cube of the series $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-$&c.

9. Find the 5th power of the series $1+\dfrac{3}{x}+\dfrac{5}{x^2}+\dfrac{7}{x^3}+\dfrac{9}{x^4}+\dfrac{11}{x^5}$

10. Find the square root of the series $\dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{1}{6}x^6+\frac{1}{8}x^8-\text{\&c.}}$

Of the Reversion of Series.

119. Reversion of Series is the finding the value of the root, or unknown quantity contained in an infinite series, by means of another infinite series, in which some other unknown quantity only is contained; the method of doing which is as follows:

RULE I. If the series consisist of all the powers of the unknown quantity, as $ax+bx^2+cx^3+dx^4+ex^5+\text{\&c}=y$, substitute the particular values of the coefficients in the given example, for a, b, c, d, &c. in the following formula, and the result will be the value of x, as required.

$$x=\frac{y}{a}-b\frac{y^2}{a^3}+(2b^2-ac)\frac{y^3}{a^5}-(5b^3-5abc+a^2d)\frac{y^4}{a^7}+(14b^2-21ab^2c+3a^2c^2+6a^2bd-a^3e)\frac{y^5}{a^9}-(42b^5-84ab^3c+28a^2bc^2+28a^2b^2d-7a^3cd-7a^3be+a^4f)\frac{y^6}{a^{11}}+\text{\&c.}$$

where it is to be observed, that if y be a large number, the series will often diverge, and $\therefore$ in that case will be of no practical use.

1. Given $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+$ &c. $=z$, to find the value of x in terms of z. Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{3}$, $d=\frac{1}{4}$, $e=\frac{1}{5}$, &c. and $y=z$, whence $\frac{1}{a}=1$, $-b=-\frac{1}{2}$, $+(2b^2-ac)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$
$-(5b^3-5abc+a^2d)=-(\frac{5}{8}-\frac{5}{6}+\frac{1}{4})=-\frac{1}{24}$
$+(14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e)=\frac{1}{120}$
$-(42b^5-84ab^3c+28a^2bc^2+28a^2b^2d-7a^2cd-7a^3be+a^4f)$
$=-(\frac{21}{16}-\frac{7}{2}+\frac{14}{9}+\frac{7}{4}-\frac{7}{12}-\frac{7}{10}+\frac{1}{6})=-\frac{1}{720}$, &c. &c. Therefore
Ans. $x=z-\frac{1}{2}z^2+\frac{1}{6}z^3-\frac{1}{24}x^4+\frac{1}{120}z^5-\frac{1}{720}z^6+$ &c.

2. Given $x-x^2+x^3-x^4+x^5-x^6+x^7-$&c. $=z$ to find the value of x in terms of z. Ans. $x=z+z^2+z^3+z^4+z^5+z^6+$&c.

3. Given $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6+\frac{1}{7}x^7-$ &c. $=z$ to find x in terms of z. Ans. $x=z+\frac{1}{2}z^2+\frac{1}{2.3}z^3+\frac{1}{2.3.4}z^4+\frac{1}{2.3.4.5}z^5$

4. Given $x-\frac{1}{2}x^2+\frac{1}{4}x^3-\frac{1}{8}x^4+\frac{1}{16}x^5-$&c.$=y$, to find the value of x in terms of y. Ans. $x=y+\frac{1}{2}y+\frac{1}{4}y^3+\frac{1}{8}y^4+\frac{1}{16}y^5+\frac{1}{32}y^6+$&c.

5. Given $y+\frac{y^2}{r}+\frac{2y^3}{3r^2}+\frac{2y^4}{3r^3}+\frac{2y^5}{3r^4}+\frac{32y^6}{45r^5}+\frac{244y^7}{315r^6}$&c. $=z$, to find the value of y in terms of z.

$$\text{Ans. } y=z-\frac{z^2}{r}+\frac{4z^3}{3r^2}-\frac{7z^4}{3r^3}+\frac{14z^5}{3r^4}-\frac{452z^6}{45r^5}+\frac{7148z^7}{315r^6}-\text{\&c.}$$

Rule II. If the series consist of the odd powers of the unknown quantity only, as $ax+bx^3+cx^5+dx^7+ex^9+$&c.$=y$, substitute the particular values of the coefficients, in any given example, for a, b, c, d, &c. in the following formula, and the result will be the value of x, as required. $x=\frac{y}{a}-b\frac{y^3}{a^4}+(3b^2-ac)\frac{y^5}{a^7}-(12b^3+a^2d-8abc)\frac{y^7}{a^{10}}+(55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e)\frac{y^9}{a^{13}}-$&c.

Where it is to be observed, that in this case, as well as in the former, the given and reverted series must be both of the same form, or otherwise they are not convertible into each other.

Given $x-\frac{1}{2.3}x^3+\frac{1}{2.3.4.5}x^5-\frac{1}{2.3.4.5.6.7}x^7+\frac{1}{2.3.4.5.6.7.89}x^9$ $-$ &c. $=z$, to find the value of x in terms of z. Here $a=1$, $b=-\frac{1}{6}$, $c=\frac{1}{120}$, $d=-\frac{1}{5040}$, $e=\frac{1}{362880}$, $y=z$.
$\therefore \frac{1}{a}=1$, $-b=\frac{1}{6}=\frac{1}{2.3}$; $(3b^2-ac)=(\frac{1}{12}-\frac{1}{120})=\frac{3}{40}=\frac{3}{2.4.5}$;
$-(12b^3+a^2d-8abc)=-(-\frac{1}{18}-\frac{1}{5040}+\frac{1}{90})=\frac{3}{2.4.6.7}$;
$+(55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e=$
$\left(\frac{55}{1296}-\frac{11}{864}+\frac{1}{3024}+\frac{1}{2880}-\frac{1}{362880}\right)=\frac{3.5.7}{2.4.6.8.9}$; therefore

$$x=z+\frac{1}{2.3}z^3+\frac{3}{2.4.5}z^5+\frac{3.5}{2.4.6.7}z^7+\frac{3.5.7}{2.4.6.8.9}z^9+\&c.$$

2. Given $x+x^3+x^5+x^7+x^9+\&c.=y$, to find the value of x in terms of y. Ans. $x=y-y^3+2y^5-5y^7+14y^9-\&c.$

3. Given $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-\frac{1}{11}x^{11}+\frac{1}{13}x^{13}-\&c.$ to find the value of x in terms of z. Ans $x=z+\frac{1}{3}z^3+\frac{2}{15}z^5+\frac{17}{351}z^7+\frac{62}{2835}z^9$

4. Given $x+\frac{1}{6}x^3+\frac{1}{24}x^5+\frac{61}{5040}x^7+\frac{277}{72576}x^9+\&c.=z$ to find x in terms of z. Ans. $x=z-\frac{1}{6}z^3+\frac{1}{24}z^5-\frac{61}{5040}z^7+\frac{277}{72576}z^9$

Rule III. When two series are equal to each other, as

$$ax+bx^2+cx^3+dx^4+\&c.=\alpha y+\beta y^2+\gamma y^3+\delta y^4+\&c.$$

and it is required to find the unknown quantity in one of them, in terms of the unknown quantity in the other, substitute the particular values of the coefficients in the given example, in the place of the known letters, in the following general formula, and the result will be the root or value required.

$$x=\alpha\frac{y}{a}+(\beta-bA^2)\frac{y^2}{a}+(\gamma-2bAB-cA^3)\frac{y^3}{a}+(\delta-bB^2-2bAC-3cA^2B-dA^4)\frac{y^4}{a}+(\varepsilon-2bBC-2bAB-3cAB^2-3cA^2C-4dA^3B-eA^5)\frac{y^5}{a}+(\Sigma-2bBD-bC^2-2bAE-CB^3-6cABC-3cA^2D-6dA^2B^2-4dA^3C-5eA^4B-fA^6)\frac{y^6}{a}\&c.,$$

where it is to be observed, that A, B, C, &c. are the coefficients of the first, second, third, &c. terms. The same form may also be rendered equally applicable to the case in which the odd powers of x and y only are concerned, by putting each of the coefficients of the even powers $=0$.

1. Given $x-\frac{1}{2}x^2+\frac{1}{2.3}x^3+\frac{1}{2.3.4}x^4+\frac{1}{2.3.4.5}x^5-\&c.=\frac{1}{2}y+\frac{1}{3}y^2+\frac{1}{4}y^3+\frac{1}{5}y^4+\frac{1}{6}y^5+\&c.$ to find x in terms of y.

Here $a=1$, $b=-\frac{1}{2}$, $c=\frac{1}{2.3}$, $d=-\frac{1}{2.3.4}$ &c.; and $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $\gamma=\frac{1}{4}$, $\delta=\frac{1}{5}$, &c.; whence $\alpha=\frac{1}{2}=A$, $\beta-bA^2=\frac{1}{3}+\frac{1}{2}\times\frac{1}{4}=\frac{11}{24}=B$

$\gamma-2bAB-cA^3=\frac{1}{4}+\frac{11}{48}-\frac{1}{48}=\frac{11}{24}=C$

$\delta-bB^2-2bAC-3cA^2B-dA^4=\frac{1381}{2880}=D$,

$\varepsilon-2bBC-2bAB-3cAB^2-3cA^2C-4dA^3B-eA^4=\frac{1543}{3840}$, &c.

Therefore $\frac{1}{2}y+\frac{11}{24}y^2+\frac{11}{24}y^3+\frac{1381}{2880}y^4+\frac{1543}{3840}y^5+\&c.$

2. Given $x+x^2+x^3+x^4+x^5+\&c.=y+\frac{1}{2}y^2+\frac{1}{4}y^3+\frac{1}{8}y^4+\frac{1}{16}y^5$, to find x in terms of y. Ans. $x=y-\frac{1}{2}y^2+\frac{1}{4}y^3-\frac{1}{8}y^4+\frac{1}{16}y^5$.

3. Given $x+\frac{x^3}{6d^2}+\frac{3x^5}{40d^4}+\frac{5x^7}{112d^6}+\&c=ny+\frac{ny^3}{6d^2}+\frac{3ny^5}{40d^4}+\frac{5ny^7}{112d^6}$ to find x in terms of y. Or $z=ny+\frac{1-n^2}{2.3d^2}Ay^2+\frac{9-n_2}{4.5d^2}By^3\&c.$

Where A, B, &c. are the foregoing terms with their proper sign.

121. If the series given be $ax+bx^2+cx^3+dx^4+ex^5+\&c.=y$, for finding the value of x in terms of y, it may be demonstrated as follows; assume $x=Ay+By^2+Cy^3+Dy^4+Ey^5+\&c.$ &c.

Then, if this value be substituted for x and its powers, in the above equation, we shall have

$$\left|\begin{array}{l} aAy \\ \\ \\ \\ \\ \\ \\ \end{array}\right.\begin{array}{l} +aB \\ +bA^2 \\ \\ \\ \\ \\ \\ \end{array}\left|\begin{array}{l} y^2+aC \\ \quad+2bAB \\ \quad+CA^3 \\ \\ \\ \\ \\ \end{array}\right|\begin{array}{l} y^3\quad+aD \\ \quad+2bAC \\ \quad+bB^2 \\ \quad+3cA^2B \\ \quad+dA^4 \\ \\ \\ \end{array}\left|\begin{array}{l} y^4+aE \\ \quad+2bAD \\ \quad+2bBC \\ \quad+3cA^2C \\ \quad+3cAB^2 \\ \quad+4dA^3B \\ \quad+eA^5 \end{array}\right|y^5+\&c.$$

And, consequently, if the coefficient of the first term be put $=1$, and each of the rest $=0$, there will arise the following results; $aA=1$; $aB+bA^2=0$; $aC+2bAB+cA^3=0$;
$aD+2bAC+bB^2+3cA^2B+dA^4=0$;
$aE+2bAD+2bBC+3cA^2C+3cAB^2+4dA^3B+eA^5=0$,

Where, finding the values of A, B, C, &c. in the first, second, third, &c. of these equations, and substituting them, as they arise in the succeeding equations, we shall have

$A=\frac{1}{a}$; $B=-\frac{b}{a^3}$; $C=\frac{2b^2-ac}{a^5}$; $D=-\frac{5b^3-5abc+a^2d}{a^7}$

$E=\frac{14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e}{a^9}$ &c. &c. which are the coefficients of y and its powers, in the reverted series.

123. When the series consists of the odd powers of x, or is of the form $ax+bx^3+cx^5+dx^7+ex^9+\&c.=y$, the value of x, in terms of y may be obtained in a similar manner, by assuming $x=Ay+By^3+Cy^5+Dy^7+Ey^9+\&c.$

For if this value be substituted for x and its powers, in the above equation, as in the former case, we shall have

$$y=\left|\begin{array}{l} aAy \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right.\begin{array}{l} +aB \\ +bA^3 \\ \\ \\ \\ \\ \\ \\ \end{array}\left|\begin{array}{l} y^3+aC \\ \quad+2bA^2B \\ \quad+cA^5 \\ \\ \\ \\ \\ \\ \end{array}\right|\begin{array}{l} y^5+AD \\ \quad+3bA^2C \\ \quad+3bAB^2 \\ \quad+5cA^4B \\ \quad+dA^7 \\ \\ \\ \\ \end{array}\left|\begin{array}{l} y^7+aE \\ \quad+3bA^2D \\ \quad+6bABC \\ \quad+bB^3 \\ \quad+5cA^4C \\ \quad+10cA^3B^2 \\ \quad+7dA^6B \\ \quad+eA^9 \end{array}\right|y^9+\&c.$$

And consequently, by putting the coefficient of the first term $=1$, and the rest $=0$, as before, there will arise the following results: $aA=1$; $aB+bA^3=0$; $aC+2bA^2B+cA^5=0$
$aD+3bA^2C+3bAB^2+5cA^4B+dA^7=0$
$aE+3bA^2D+6bABC+bA^3+5cA^4C+10cA^3B^2+7dA^6B+eA^9=0$;
where, finding the values of A, B, C, &c. in the first, second,

third &c. of these equations, as in the former case, and then substituting them, as they arise, in the succeeding equations, we shall have $A=\frac{1}{a}$; $B=-\frac{b}{x^4}$; $C=\frac{3b^2-ac}{a^7}$; $D=-\frac{12b^3+a^2d-8abc}{a^{10}}$; $E=\frac{55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e}{a^{13}}$, &c. which are the coefficients of y and its powers, in the reverted series, for this case.

124. When there are two series, consisting of the successive powers of x and y, as $ax+bx^2+cx^3+dx^4+\&c.=\alpha y-\beta y^2+\gamma y^3+\delta y^4$

Let there be assumed, as in the instances before given, $x=Ay+By^2+Cy^3+Dy^4+Ey^5+\&c.$ Then if this value be substituted for x and its powers, in the left hand member of the proposed equation, we shall have $\alpha y+\beta y^2+\gamma y^3+\delta y^4+\varepsilon y^5+\&c.$ $=aAy+(aB+bA^2)y^2+(aC+2bAB+cA^3)y^3+(aD+2bAC+bB^2+3cA^2B+dA^4)y^4+(aE+2bAD+2bBC+3cA^2C+3cAB^2+4dA^3B+eA^5)y^5+\&c.$ and, consequently, by equating the coefficients of the homologous terms, there will arise the following results: $aA=\alpha$, $aB+bA^2=\beta$, $aC+2bAB+cA^3=\gamma$, $aD+2bAC+bB^2+3cA^2B+da^4=\delta$, $aE+2bAD+2bBC+3cA^2C+3cAB^2+4dA^3B+eA^5=\varepsilon$, where, finding the values of A, B, C, &c. as in the former cases, and substituting them, as they arise, in the succeeding equations, we shall have $A=\frac{\alpha}{a}$, $B=\frac{\beta-bA^2}{a}$; $C=\frac{\gamma-2bAB-cA^3}{a}$;

$D=\frac{\delta-bB^2-2bAC-3cA^2B-dA^4}{a}$;

$E=\frac{\varepsilon-2bBC-2bAD-3cAB^2-3cA^2C-4dA^3B-eA^5}{a}$, &c. &c.

which are the coefficients of the terms of the series that give the value of x in terms of y.

Of Permutations and Combinations.

Permutation is a change in the order or position of any number of things, so that no two parcels of them shall be situated alike.

Thus, the two letters *a*, *b*, admit of two permutations, viz. *ab* and *ba*.

The three letters, *a*, *b*, *c*, admit of six permutations, viz. *abc*, *bac*, *cab*, *acb*, *bca*, *cba*.

By combination of quantities is meant the different collections that can be formed of them, without regarding the order or position in which the quantities are placed.

Thus, the number of combination which can be made of three things, *a*, *b*, *c*, taken two and two together, is 3; viz. *ab*, *ac*, *bc*.

The number of combinations of four things, *a*, *b*, *c*, *d*, taken two and two together, is 6; viz. *ab*, *ac*, *ad*, *bc*, *bd*, *cd*. If taken three at a time, the number of v *acd*,combinations is 4; iz. *abc*, *bcd*, *abd*.

Theorem 1. The number of permutations that can be formed out of (n) quantities taken two and two together, is $n.(n-1)$; taken three and three together, the number is $n.(n-1).(n-2)$.

Let a, b, c, d, e, &c. be the (n) quantities, which are to be taken two at a time. Now, by placing a before each of the other ($n-1$) quantities, we form ($n-1$) permutations. Also, by placing b before each of the remaining ($n-1$) quantities, we form ($n-1$) permutation; and this process can be repeated (n) times, therefore in all, we obtain, $n(.n-1)$ permutations.

Again, of ($n-1$) quantities, when two are taken at a time, we have $(n-1).(n-2)$ permutations, as appears from what has been already said; and since a can be prefixed to each of these quantities, there may arise $(n-1).(n-2)$ permutations in which a stands first; and this process may be repeated (n) times, ∴ upon the whole we obtain $n(n-1).(n-2)$ permutations when three quantities are taken at a time.

By following the same method, it appears, that in (n) things, if r of them be taken together, there are $n.(n-1).(n-2).(n-3)\ldots.(n-r+1)$ permutations.

Theorem 2. In (n) quantities, containing (p) quantities of one sort, (q) quantities of another sort, &c. the permutations will be

$$\frac{n.(n-1).(n-2)\ \ldots.3.2.1}{1.2.3\ldots.p\times1.2.3\ldots.q\times1.2.3\ldots.r\&c.}$$

Now when the quantities are all different from one another, the number of permutations $=n.(n-1).(n-2)\ldots.3.2.1$, taking ($n$) at a time, by Theorem 1; but if any quantity recur twice, the number of permutations is diminished 2.1 times:* and if any quantity recur thrice, the number of permutations is diminished 3.2.1 times: and in general if there be (p) quantities alike, the number of permutations is diminished $1.2.3\ldots.p$ times. The same may be shown with respect to the (q) quantities. Hence the number of permutations which can be formed

$$=\frac{n(.(n-1).(n-2)\ldots\ldots..3.2.1}{1.2.3\ldots.p\times1.2.3\ldots.q\times1.2.3\ldots.r\&c.}$$

1. Required the number of variations or permutations that can be formed of the letters in the word "Bacchanalia."

Here the letter c recurs twice, a four times, and there are 11 letters in all; ∴ the number of variations required

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* Since each combination of two quantities admits of 1.2 permutations, and each combination of three quantities admits of 1.2.3 permutations, ∴ the quantities 1.2, 1.2.3, &c. become divisors of that number which would represent the number of permutations if the quantities were all different.

$$=\frac{11.10.9.8.7.6.5.4.3.2.1}{1.2\times1.2.3.4}=831600.$$

2. How many different numbers can be made of the following digits; 1, 2, 2, 7, 7, 7, 5, 5, 5, 5? Ans. 12600.

3. How many permutations may be made of the letters in the word "examination." Ans. 4989600.

Theorem 3. The number of combinations that can be formed out of (n) things, taken two and two together, is $\frac{1}{2}\{n.(n-1)\}$; when taken three and three together $\frac{n(n-1)\cdot(n-2)}{1.2.3}$.

Now since each combination *ab* admits of two permutations, ∴ when the quantities are taken two at a time, there will be 1.2 times as many permutations as combinations. Now by Theorem 1, the number of permutations of (n) things, taken two at a time, $=n.(n-1)$; hence the number of permutations = 1·2 times the number of combinations; ∴ the number of combinations $=\frac{n.(n-1)}{1.2}$.

Again, by Theorem 1, there are $n.(n-1).(n-2)$ permutations in (n) things, taken three at a time; but each combination of 3 things admits of 1.2.3 permutations, ∴ $n.(n-1).(n-2)=1.2.3$ times the number of combinations, i. e. the number of combinations $=\frac{n.(n-1).(n-2)}{1.2.3}$. By following the same method, it appears that the number of permutations of (n) quantities, taken (r) at a time, $=n(n-1).(n-2)\ldots(n-r+1)$. Also each combination of (r) things admits of $1.3.3.4\ldots.r$ permutations; ∴ the number of permutations $=1.2.3\ldots.r$ times the number of combinations; hence the number of combinations in (n) quantities, taking r of them at a time, $=\frac{n.(n-1).(n-2)\ldots.(n-r+1)}{1.2.3\ldots.r}$

Theorem 4. The total number of combinations which can be formed of (n) things taken, one at a time, two at a time, &c. $=2^n-1$.

First let the quantities be taken one at a time, then the number of combinations is ....n. If taken two at a time, the number is $n.\frac{n-1}{2}$. If taken 3 at a time, the number is $n.\frac{(n-1).(n-2)}{1\cdot2\cdot3}$.

If taken four at a time, the number is $n.\frac{(n-1).(n-2)\cdot(n-3)}{1.2.3.4}$

$$\therefore \text{the total number} = n+\frac{n.(n-1)}{1.2}+\frac{n.(n-1).(n-2)}{1.2.3}+\&c.$$
$$=(1+1)^n-1=2^n-1.$$

Investigation of the Rules of Compound Interest.

Let $a =$ annuity, rent, or pension, $n =$ number of times that interest is to be paid for the annuity or sum lent, $r =$ rate of interest of 1 dollar for 1 time, $m =$ amount of the annuity, or sum lent, for n times, at r interest, $p =$ principal, sum used, or present worth of a sum before it is due, (of compound interest.) First in amounts let $q = 1 + r =$ amount of one dollar for 1 time. Now $a =$ last year's amount, and $1 : q :: a : aq =$ last year but 1 amt.
$1 : q :: aq : aq^2 =$ last but 2 year's amount.
$1 : q :: aq^2 : aq^3 =$ last but 3, and so on to $aq^{n-1} =$ first year's amount; therefore $a + aq + aq^2 + aq^3$, &c., $+ aq^{n-1} = m$; but $a : aq :: m - aq^{n-1} : m - a$. then $m - a = mq - aq^n$, or $m - a = m + mr - aq^n$, $\therefore aq^n - a = mr$, then put $A = q^n$, $(A-1)a = mr$, $\therefore a = \frac{mr}{*A-1}$; $m = \frac{(A-1)a}{r}$; $r = \frac{(A-1)a}{m}$.

2d. In discounts, $q : 1 :: a : \frac{a}{q} =$ first year's present worth, and $q : 1 :: \frac{a}{q} : \frac{a}{q^2} =$ 2d year's present worth; $q : 1 :: \frac{a}{q^2} : \frac{a}{q^3} =$ 3d year, and so on $\frac{a}{q^n} =$ nth year's present worth, therefore I have $\frac{a}{q} + \frac{a}{q^2} + \frac{a}{q^3} +$ &c. to $\frac{a}{q^n} = p$; but $\frac{a}{q} : \frac{a}{q^2} :: p - \frac{a}{q^n} : p - \frac{a}{q}$; (Euc. 12, V.) $\therefore \frac{pq^n - a}{q^n} = pq - a$, or $\frac{pq^n - a}{q^n} = p + pr - a$; $\therefore aq^n - a = prq^n$, or $(A-1)a = prA$; $\therefore A = \frac{a}{a-rp}$; hence $(A-1)a = mr = prA$; $\therefore A = \frac{mr+a}{a} = \frac{a}{a-rp} = \frac{m}{p} = (1+r)^n$.

Case 1. Given a, m, and p, to find r.

Since $A = \frac{m}{p}$; $\therefore r = \frac{(A-1)a}{m}$.

$*A = \frac{mr+a}{a}$.

Case 2. Given p, m, and n, to find r.

Since $(r+1)^n = \frac{m}{p}$; $\therefore r = \left(\frac{m}{p}\right)^{\frac{1}{n}} - 1$.

Case 3. Given a, m, and n, to find r.

Since $(A-1)a = mr$; therefore $\frac{(A-1)}{nr} = \frac{m}{na} = \frac{(1+r)^n - 1}{nr}$; now $(1+r)^n = 1 + nr + n\left(\frac{n-1}{2}\right)r^2 + n\left(\frac{n-1}{2}\right) \times \left(\frac{n-2}{3}\right)r^3$, &c.; $\therefore \frac{m}{na} = 1 + \frac{n-1}{2}r + \frac{n-1}{2} \times \frac{n-2}{3}r^2$, &c., and $\left(\frac{m}{an}\right)^{\frac{2}{n-1}} = \left(1 + \frac{n-1}{2}r + \frac{n-1}{2} \times \frac{n-2}{3}r^2\right)^{\frac{2}{n-1}}$; which by the binomial theorem will become $=1+r+\frac{n+1}{12}r^2$ nearly.

Let $D=\left(\frac{m}{an}\right)^{\frac{2}{n-1}}=1+r+\frac{n+1}{12}r^2$; then $r^2+\frac{12r}{n+1}=(D-1)\times\frac{12}{n+1}$; let $2E=\frac{12}{n+1}$; $r+E=(\sqrt{\{(2\times[D-1]+E)\times E\}}=F$; therefore $r=F-E$. In this solution, 1st find $D=\left(\frac{m}{na}\right)^{\frac{2}{n-1}}$; 2d, find $E=\frac{6}{n+1}$; 3d, find $F=\sqrt{\{2(D-1)+E\}\cdot E}$; 4th, find $r=F-E$.

Case 4. Given a, p, and n, to find r.

Since $(A-1)a=Apr$, $\therefore\ \frac{p}{na}=\frac{A-1}{Anr}=\frac{1-A^{-1}}{nr}=\frac{1-(1+r)^{-n}}{nr}$; now $(1+r)^{-n}=1-nr+n\times\frac{n+1}{2}r^2-n\times\frac{n+1}{2}\times\frac{n+2}{3}r^3+$ &c.; therefore $\frac{p}{na}=1-\frac{n+1}{2}r+\frac{n+1}{2}\times\frac{n+2}{3}r^2$, nearly; then $\left(\frac{p}{na}\right)^{-\frac{2}{n+1}}=\left(1-\frac{n+1}{2}r+\frac{n+1}{2}\times\frac{n+2}{3}r^2\right)^{-\frac{2}{n+1}}$; which, by binomial theorem, will become $=1+r-\frac{n-1}{12}r^2$, nearly. Now, $\left(\frac{p}{na}\right)^{-\frac{2}{n+1}}=\left(\frac{na}{p}\right)^{\frac{2}{n+1}}$; let $G=\left(\frac{na}{p}\right)^{\frac{2}{n+1}}=1+r-\frac{n-1}{12}r^2$; therefore $r^2-\frac{12}{n-1}r=-(G-1)\times\frac{12}{n-1}$. Let $2H=\frac{12}{n-1}$, then $r-H=(\sqrt{\{H-2(G-1)\}\times H}=)K$; $\therefore r=H-K$.

Case 5. Given a, n and r, to find m.

Since $(A-1)a=mr$, therefore $m=\frac{(A-1)a}{r}$.

Case 6. Given p, n, and r, to find m.

Since $A=\frac{m}{p}$, $\therefore\ m=pA$.

Case 7. Given a, p, and r; find m.

Since $\frac{m}{p}=\frac{a}{a-rp}$; $\therefore\ m=\frac{ap}{a-rp}$.

Case 8. Given a, p, and n, to find m.

Find $G=\left(\frac{na}{p}\right)^{\frac{2}{n+1}}$; $H=\frac{3}{\frac{1}{2}(n-1)}$; $K=\sqrt{\{H-(2G-1)\}\times H}$. Now $H-K=r$, by the 4th, but $(r+1)^n=\frac{n}{p}$; $\therefore m=(r+1)^n\times p$.

Case 9. Given a, n, and r, to find p.

Since $(A-1)a=prA$; therefore $p=\frac{(A-1)\times a}{Ar}$

Case 10. Given m, n, and r; required p.

Since $A = \frac{m}{p}$; $\therefore p = \frac{m}{A}$, … $\left\{ \begin{array}{l} p(1+r)^n = m \text{ in Logarithms} \\ P = M - n \times L(1+r). \end{array} \right.$

Case 11. Given a, m, and r, to find p.

Since $\frac{mr+a}{a} = \frac{m}{p}$; $\therefore p = \frac{ma}{mr+a}$.

Case 12. Given a, m, and n, to find p.

Find $D = \left(\frac{m}{na}\right)^{\frac{2}{n-1}}$; $E = \frac{2}{\frac{1}{2}(n+1)}$; $F = \sqrt{\{2(D-1)+E\} \times E}$.

Now $F - E = r$, by 3d, but $(1+r)^n = \frac{m}{p}$; $\therefore p = \frac{m}{(1+r)^n}$.

Case 13. Given p, n, and r, to find a.

Since $(A-1)a = prA$; $\therefore a = \frac{prA}{A-1}$.

Case 14. Given p, m, and n, to find a.

Since $A = (r+1)^n = \frac{m}{p}$; $\therefore r+1 = \left(\frac{m}{p}\right)^{\frac{1}{n}}$. Hence $A-1$, and $B-1 = r$, are known; $\therefore a = \frac{m \times (B-1)}{A-1}$.

Case 15. Given m, n, and r, to find a.

Since $(A-1)a = mr$, therefore $a = \frac{mr}{A-1}$.

Case 16. Given p, m, and r, to find a.

Since $A = \frac{m}{p}$, $A-1$ is given, $\therefore a = \frac{mr}{A-1}$.

Case 17. Given m, p, and r, to find n.

Put L for the logarithm, and L′ for the arith. comp. of a log.

Since $(1+r)^n = \frac{m}{p}$, therefore $n = \frac{L.m + L'.p}{L.(r+1)} = \frac{M - p}{L(1+r)}$.

Case 18. Given a, p, and r, to find the value of n.

Since $(1+r)^n = \frac{a}{a-pr}$, therefore $n = \frac{L.a - L.(a-pr)}{L.(r+1)}$.

Case 19. Given a, p, and m, to find n.

Since $A = (r+1)^n = \frac{m}{p}$; then $L.(r+1)^n = L.m - L.p = L.A$. Hence A, and $A-1$, are known; Also $L.(A-1) + L.a + L'.m = L.(B-1)$ by Case 14th hence $r = B - 1$ and $r+1$, are known therefore $n = \frac{L.A.}{L.(r+1)}$.

Case 20. Given a, m, and r, to find n.

Since $A = \frac{mr+a}{a} = (1+r)^n$; $\therefore n = \frac{L.(mr+a) - L.a}{L.(1+r)}$.

TABLES OF COMPOUND INTEREST AND ANNUITIES.

Present Value of $1 for 21 years, discounting at Comp. Interest.

Yrs.	at 3 per ct.	3½ per ct.	4 per ct.	4½ per ct.	5 per ct.	6 per ct.
1	.9708738	.9061836	.9615385	.9569378	.9523809	.943396
2	.9425959	.9335107	.9245562	.9157299	.9070295	.889996
3	.9151417	.9019427	.8889964	.8762966	.8638376	.839619
4	.8884870	.8714422	.8548042	.8385613	.8227025	.792094
5	.8626088	.8419732	.8219271	.8024511	.7835262	.747258
6	.8374843	.8135006	.7903145	.7678957	.7462154	.704961
7	.8130915	.7859910	.7599178	.7348285	.7106813	.665057
8	.7894092	.7594116	.7306902	.7031851	.6768394	.627412
9	.7664167	.7337310	.7025867	.6729044	.6446089	.591898
10	.7440939	.7089188	.6755642	.6439277	.6139133	.558395
11	.7224213	.6849457	.6495809	.6161988	.5846793	.526788
12	.7013799	.6617833	.6245971	.5896639	.5568374	.496969
13	.6809513	.6391041	.6005741	.5642716	.5303214	.468839
14	.6611178	.6177818	.5774751	.5399729	.5050679	.442301
15	.6418619	.5968906	.5552645	.5167204	.4810171	.417265
16	.6231669	.5767059	.5339082	.4944693	.4581115	.393646
17	.6050164	.5572038	.5133733	.4731764	.4362967	.371364
18	.5873946	.5383611	.4936281	.4528004	.4155207	.350344
19	.5702860	.5201557	.4746424	.4433018	.3957340	.330513
20	.5536758	.5025659	.4566870	.4146429	.3768895	.311805
21	.5375493	.5855709	.4388336	.3967874	.3589424	.294155

Amount of $1 Annuity for 21 years, calculated at Comp. Interest.

1	1.0000000	1.00000	1.00000	1.	1.	1.000000
2	2.0300000	2.03500	2·04000	2.04500	2.05000	2.060000
3	3.0909000	3.10623	3.12160	3.13703	3.15250	3.183600
4	4.1836270	4.21494	4.24646	4.27819	4.31013	4.374616
5	5.3091358	5.36247	5.41632	5.47071	5.52563	5.637092
6	6.4684099	6.55015	6.63298	6.71689	6.80191	6.975316
7	7.6624622	7.77941	7.89829	8.01915	8.14204	8.393838
8	8.8923361	9.05169	9.21423	9.38001	9.54911	9.897468
9	10.1591061	10.36880	10.58280	10.80211	11.02656	11.491319
10	11.4638793	11.73139	12.00611	12.28821	12.57789	13.180793
11	12.8077957	13.14199	13.48635	13.84118	14.20679	14.791643
12	14.1920296	14.60196	15.02581	15.46403	15.91713	16.869941
13	15.6177905	16.11303	16.62684	17.15991	17.71298	18.882138
14	17.0863242	17.67699	18.29191	18.93211	19.59863	21.015066
15	18.5989139	19.29568	20.02359	20.78405	21.57856	23.275970
16	20.1568813	20.97103	21.82453	22.71934	23.65749	25.672528
17	21.7615877	22.70502	23.69751	24.74171	25.84037	28.212880
18	23.4144354	24.49969	25.64541	26.85508	28.13238	30.905653
19	25.1168684	26.35718	27.67123	29.06356	30.53900	33.759892
20	26.8703745	28.27968	29.77808	31.37142	33.06595	36.785591
21	28.6764857	30.26947	31.96920	33.78314	35.71925	39.992727

A TABLE of the present worth of $1 *annuity for* 21 *years.*

Yrs	3 per ct.	3½ per ct.	4 per ct.	4½ per ct.	5 per ct.	6 per ct.
1	0.970874	0.966184	0.961539	0.956938	0.952381	0.9524
2	1.913470	1.899694	1.886095	1.872668	1.859410	1.8334
3	2.828611	2.801637	2.775091	2.748964	2.723248	2.6730
4	3.717098	3.673079	3.629895	3.587526	3.545951	3.4651
5	4.579707	4.515052	4.451822	4.389977	4.329477	4.2124
6	5.417191	5.328553	5.242137	5.157873	5.075692	4.9173
7	6.230283	6.114544	6.002055	5.892701	5.786373	5.5824
8	7.019692	6.873956	6.732745	6.595886	6.463213	6.2098
9	7.786109	7.607687	7.435332	7.268791	7.107822	6.8017
10	8.530203	8.316605	8.110896	7.912718	7.721735	7.3601
11	9.252624	9.001551	8.760477	8.528917	8.306414	7.8869
12	9.954004	9.663334	9.385074	9.118581	8.863252	8.3838
13	10.634955	10.302739	9.985647	9.682852	9.393573	8.8527
14	11.296073	10.920520	10.563122	10.222825	9.898641	9.2950
15	11.937935	11.517411	11.118387	10.739546	10.379658	9.7123
16	12.561102	12.094117	11.652296	11.234015	10.837770	10.1059
17	13.166119	12.651321	12.165669	11.707191	11.274066	10.4773
18	13.753513	13.189682	12.659297	12.159992	11.689587	10.8276
19	14.323799	13.709837	13.133939	12.593294	12.085321	11.1581
20	14.877475	14.212403	13.590326	13.007937	12.462210	11.4699
21	15.415024	14.697974	14.029160	13.404724	12.821153	11.7641

A TABLE of the amount of which $1 *will purchase for* 21 *years.*

1	1.0300000	1.0350000	1.0400000	1.045000	1.0500000	1.06000
2	.5226108	1.5264005	.5301961	.5339976	.5378049	.54543
3	.3535304	.3569342	.3603485	.3637734	.3672086	.37411
4	.2690271	.2722511	.2754901	.2787437	.2820118	.28859
5	.2183546	.2214814	.2246271	.2277916	.2309748	.23739
6	.1845975	.1876682	.1907619	.1938784	.1970175	.20336
7	.1605064	.1635445	.1666096	.1697015	.1728198	.17913
8	.1424564	.1454767	.1485278	.1516097	.1547218	.16103
9	.1284339	.1314460	.1344930	.1375745	.1406901	.14702
10	.1172305	.1202414	.1232909	.1263788	,1295046	.13586
11	.1080775	.1110920	.1141490	.1172482	.1203889	.12079
12	.1004621	.1034840	.1065522	.1096662	.1128254	.11927
13	.0940295	.0970616	.1001437	.1032754	.1064558	.11296
14	.0885263	.0915707	.0946690	.0978203	.1010240	.10758
15	.0837666	.0868251	.0899411	.0931138	.0963423	.10299
16	.0796109	.0826848	.0858200	.0890154	.0922699	.09895
17	.0759535	.0790431	.0821985	.0854176	.0886991	·09544
18	.0727087	.0758168	.0789933	.0822369	.0855462	.09235
19	.0698139	.0729403	.0761386	.0794073	.0827450	.08962
20	.0672157	.0703611	.0735818	.0768761	.0802426	.08718
21	.0648718	.0680366	.0712801	.0746006	.0779961	.08500